THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and forty-seventh regular meeting of the Society was held in New York City on Saturday, February 26, 1910, extending through the usual morning and afternoon sessions. The following twenty-eight members were present:

Professor G. D. Birkhoff, Professor C. L. Bouton, Professor E. W. Brown, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Professor L. P. Eisenhart, Professor Peter Field, Professor T. S. Fiske, Dr. C. C. Grove, Professor J. I. Hutchinson, Dr. L. C. Karpinski, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor J. H. Maclagan-Wedderburn, Mr. A. R. Maxson, Mr. H. H. Mitchell, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Dr. W. M. Strong, Dr. Elijah Swift, Professor E. B. Van Vleck, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White.

Ex-President W. F. Osgood occupied the chair at the morning session, Vice-President J. I. Hutchinson at the afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. E. S. Allen, Berkshire School, Sheffield, Mass.; Mr. B. A. Bernstein, University of California; Mr. G. W. Evans, Charlestown High School, Boston, Mass.; Mr. C. E. Flanagan, Wheeling, W. Va.; Mr. C. E. Githens, Wheeling, W. Va.; Mr. J. S. Mikesh, University of Minnesota; Professor G. P. Paine, University of Minnesota; Mr. W. L. Putnam, Boston, Mass.; Mr. V. M. Spunar, Pittsburg, Pa. Nine applications for membership in the Society were received.

Committees were appointed by the Council to arrange for the coming summer meeting and to report on the subject of the publication of the Princeton Colloquium Lectures.

The following papers were read at this meeting:

(1) Professor G. D. Birkhoff: "A simplified treatment of the regular singular point."

(2) Professor G. D. Birkhoff: "Some oscillation and comparison theorems."

(3) Professor P. F. Smith: "On osculating bands of surface-element loci."

(4) Professor EDUARD STUDY: "Die natürlichen Gleichungen der analytischen Curven im euklidischen Raume."

(5) Professor G. A. MILLER: "Addition to Sylow's theorem."

(6) Professor PETER FIELD: "On the circuits of a plane curve."

(7) Professor C. L. BOUTON: "Examples of transcendental one-to-one transformations."

(8) Professor JACOB WESTLUND: "On the fundamental number of the algebraic number field $k(\sqrt[p]{m})$."

(9) Professor L. P. EISENHART: "Surfaces with isothermal representation of their lines of curvature, and their transformations (second paper)."

(10) Professor EDWARD KASNER: "Isothermal nets."

(11) Dr. ARTHUR RANUM: "On the principle of duality in spherical geometry."

(12) Dr. O. E. GLENN: "On multiple factors of ternary and quaternary forms: applications to resolution of rational fractions."

In the absence of the authors the papers of Professor Smith, Professor Study, Professor Miller, Professor Westlund, Dr. Ranum, and Dr. Glenn were read by title. Professor Study's paper will appear in the July number of *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. This note by Professor Birkhoff is to appear in an early number of the *Transactions*. The theorems of Fuchs concerning the solutions of ordinary linear differential equations in the vicinity of a regular singular point are proved by a very short analysis which does not involve the substitution of an infinite series in the equation.

2. In a second note Professor Birkhoff derived oscillation and comparison theorems for ordinary linear equations of the third order analogous to the well-known one for second order equations. This paper will be offered to the *Annals of Mathematics*.

3. Professor Smith's paper is a continuation of that presented by the author at the summer meeting of 1909. The new matter consists chiefly in the discussion of cubic bands in space, that is, unions of ∞^1 surface elements whose points lie on a skew cubic and whose planes envelope a quadric cone. The vertex of the cone must lie on the cubic. If this cubic is

required to pass through three fixed points, then a cubic band is uniquely determined by two united surface elements. The point of contact with the theorems established in the former paper lies in this last statement. The cubic bands which osculate the characteristic bands of a partial differential equation of the first order osculate also the characteristic bands of a second differential equation. Other theorems analogous to those proved in the first paper hold for cubic bands also. Incidentally, a simple contact transformation appears, under which parabolic bands become cubic bands, vertical parabolas become fournodal cubic surfaces, and straight lines become quadric cones.

5. According to Sylow's theorem the Sylow subgroups of order p^m are transformed according to a transitive substitution group. Professor Miller's paper deals with the question whether this transitive group is primitive or imprimitive. His main theorem may be stated as follows: If a group G contains more than one Sylow subgroup of order p^m and if one such subgroup has no more than p^a substitutions in common with any of the others, then G cannot transform these subgroups according to an imprimitive group unless their number is of the form $(1 + kp^{m-a})(1 + lp^{m-a})$, where both k and l are greater than zero. As a direct corollary we have that a group cannot transform its Sylow subgroups of order p^m according to an imprimitive group unless the number of these subgroups is of the form $(1 + kp)(1 + lp)$ where both k and l are integers greater than 0. For instance, if the order of G is divisible by 7, its Sylow subgroups of order 7^m cannot be transformed according to an imprimitive group unless their number is at least 64, and if it exceeds this number it must be at least 120, etc. This theorem is frequently very useful to determine whether a given substitution group is primitive.

6. It is known that there exist curves for every order n, $p = 0$ or 1, composed of a single circuit of index $n - 2$. In case $p = 1$ the curve may also have a simple oval (C. A. Scott, *Transactions*, volume 3). Professor Field's paper is devoted to a proof that there exist for every order n, $p = r$ $(r = 1, 2, 3, \cdots, n - 2)$ curves composed of r circuits, the sum of whose indices is $n - 2$. There may also be in addition a simple oval.

7. In this paper Professor Bouton gives a method for constructing a general class of transcendental one-to-one transfor-

mations in n variables, $n>1$. To his knowledge no examples of such transformations have heretofore been given. A simple example is

$$u = x + e^{y+\sin x}, \quad v = y + \sin x,$$

with the inverse

$$x = u - e^v, \quad y = v - \sin(u - e^v).$$

The method also gives classes of transcendental one-to-one contact transformations in n variables.

8. In Professor Westlund's paper the algebraic number field $k(\sqrt[p]{m})$ generated by the real pth root of a number m, where p is any odd prime, is discussed. An integral basis and the fundamental number of k are determined.

9. In a former paper with the same title (*Transactions*, volume 9) Professor Eisenhart established a transformation of surfaces S with isothermal spherical representation of their lines of curvature into surfaces of the same kind, such that upon a surface S and a transform S' the lines of curvature correspond, and S and S' constitute the envelope of a two-parameter family of spheres. This transformation was established by means of the Thybaut transformation of minimal surfaces and the transformation of partial differential equations of the second order due to Moutard. In the present paper transformations of this kind are established in a more simple and fundamental manner by determining the W-congruences (that is, congruences upon whose focal surfaces the asymptotic lines correspond) which are of such a character that the Lie line-sphere transformation converts the focal surfaces of the congruence into surfaces, S and S', with isothermal spherical representation of their lines of curvature. At the same time, S and S' are then necessarily the envelope of a two-parameter family of spheres. This method seems to give a more satisfactory reason for the existence of such transformations. The analysis is very much simpler in this case, and one obtains almost immediately a "theorem of permutability" for transformations of this kind.

10. Professor Kasner obtains the conditions which must be satisfied by two analytic curves intersecting orthogonally in order that they may be regarded as members of an isothermal net; that is, in order that they may be conformally transformable into orthogonal straight lines. The conditions are expressed

as an infinite set of relations between the coefficients of the power series representations of the given curves. All the coefficients of one series and half the coefficients of the other series may be taken at random, the other half then being determined by the general relations. Geometric interpretations are obtained by introducing the successive derivatives of the radii of curvature with respect to arc.

11. In the geometry of the surface of a sphere the principle of duality applies to metrical as well as projective properties. The consequences of this elementary fact do not seem to have been utilized much in the past. In this paper Dr. Ranum shows how the duality between distance and angle, between arc and area, between rolling and sliding, immediately leads to some interesting new theorems. For example, any continuous spherical movement can be regarded as due to the sliding (without rolling) of one spherical curve on another; the locus of a point lying on a variable tangent to a curve at a distance of $\frac{1}{2}\pi$ from the point of tangency is the envelope of the polar of the center of curvature at a variable point of the same curve; the area swept out by a moving tangent to one of two polar curves is equal to twice the length of the corresponding arc of the other.

12. In a paper published in the *American Journal of Mathematics*, volume 32, number 1, Dr. Glenn developed the theory of the decomposition of ternary and quaternary forms into simple quadratic factors. In the present paper this theory is extended to the cases where multiple factors occur. The multiple linear factors and multiple quadratic factors of a form are completely determined, save as to the solution of certain ordinary algebraical equations of higher degree.

In a footnote in the before-mentioned paper a new partial fraction theorem was announced, the unique feature of which was that the numerators of those partial fractions of b_x^{m-1}/a_x^m which have simple quadratic denominators were determined by ordinary differentiation and application of the Aronhold operator $(b \cdot \partial/\partial a)$. This theory is brought to a degree of completion in the present paper. Differential operators are developed by which are determined the numerators of the fractions corresponding to multiple quadratic factors of a_x^m. This theory is also extended to include the case where b_x^{m-1} and a_x^m are ternary forms.

F. N. COLE,
Secretary.

THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE seventeenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, on Saturday, February 26, 1910. The following members were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor C. A. Noble, Mr. E. W. Ponzer, Professor T. M. Putnam, Dr. H. W. Stager.

Professor Blichfeldt occupied the chair. A committee consisting of Professors Allardice, Haskell, and Lehmer was appointed to draft resolutions in memory of the late Professor Irving Stringham.

The following papers were read at this meeting:

(1) Professor L. M. HOSKINS: "The strain of an elastic sphere of which the density is a function of the distance from the centre."

(2) Professor D. N. LEHMER: "A list of primes."

(3) Professor W. A. MANNING: "A note on Bochert's theorem."

(4) Professor C. A. NOBLE: "Characteristics of two partial differential equations of the first order."

(5) Mr. E. W. PONZER: "The principles of the calculus as applied in technical courses."

Abstracts of the papers are given below in order as numbered in the foregoing list:

1. The problem solved in the paper of Professor Hoskins is that of the strain of an elastic sphere of which the density is any known function of the distance from the center, the bodily forces having a potential which is a known function of the coordinates of position. The only restriction on the potential function is that it is developable in a series of spherical surface harmonics; an analogous restriction must hold regarding the surface stresses. General formulas are deduced for the elastic displacements, and the solution is given in full for the case in which the potential is a spherical solid harmonic of the second degree while the surface is free from stress. In this case the

surface values of the ellipticity e and angular displacement α are found to be expressible in simple form. Computations for the case of Laplace's law of density show that, for given values of the modulus of rigidity and total mass, e is about 11 per cent. less and α about 22 per cent. less than in the case of uniform density, these results being nearly independent of the compressibility.

2. The list of primes upon which Professor Lehmer is engaged is intended to extend to the limit of the factor tables recently published by the Carnegie Institution of Washington. The list is already completed for the primes contained in the 1st, 2d, 7th, 8th, 9th, and 10th millions. The table is to contain 133 pages, each with 5,000 entries. There are fifty columns, each containing 100 lines. As the table stands, it will be only a few moments' work to obtain the number of primes between any two given limits, or the rank of any given prime in the series of primes. The number of primes in each successive thousand is checked against the counts made by Glaisher (see his Introduction to the factor table for the sixth million) and the number in each successive fifty thousand is compared with the results of Bertelsen's computations (see *Acta Mathematica*, volume 17, 1893). No discrepancies with the latter have as yet appeared and the frequent differences with Glaisher are all due to the presence of errors in the tables from which he made his count.

3. In this note Professor Manning shows that if the substitutions of degree u in a 4-fold transitive group of class u and degree n (not alternating) are all of order 2, then $u > \frac{1}{2}n$.

4. Professor Noble deduces, after Hilbert, the ordinary differential equations of the characteristics of two partial differential equations of first order, shows how these suffice to obtain the general solution of the given equations, and constructs the corresponding problem in the calculus of variations.

5. Mr. Ponzer's paper presents a study of the various principles of the calculus as applied in the principal technical courses, both qualitatively and quantitatively; the nature and frequency of the principles applied is discussed, with suggestions as to where emphasis should be placed, and what should be omitted, in a course in the calculus.

C. A. NOBLE,
Secretary of the Section.

AN APPLICATION OF THE NOTIONS OF "GENERAL ANALYSIS" TO A PROBLEM OF THE CALCULUS OF VARIATIONS.

BY PROFESSOR OSKAR BOLZA.

(Read before the Chicago Section of the American Mathematical Society, April 8, 1910.)

THE object of the following note is to give an illustration of the unifying power of Professor E. H. Moore's methods of "General Analysis" * by showing that a certain theorem of the calculus of variations and a certain theorem of analytic geometry are special cases of one and the same theorem of general analysis.

The theorem of the calculus of variations is the so-called fundamental lemma for isoperimetric problems,† viz.,

THEOREM I. *"If*

$$(1) \qquad \mu_0(\eta) \equiv \int_{x_1}^{x_2} [M_0(x)\eta(x) + N_0(x)\eta'(x)]\,dx = 0$$

for all functions $\eta(x)$ which are (a) of class C' on $[x_1x_2]$, (b) vanish at x_1 and x_2, and (c) satisfy the m conditions

$$(2) \qquad \mu_i(\eta) \equiv \int_{x_1}^{x_2} [M_i(x)\eta(x) + N_i(x)\eta'(x)]\,dx = 0 \qquad (i = 1, 2, \cdots, m),$$

then there exist m constants $c_1, c_2, \cdots, c_m$ such that

$$(3) \qquad \mu_0(\eta) + c_1\mu_1(\eta) + c_2\mu_2(\eta) + \cdots + c_m\mu_m(\eta) = 0$$

for all functions $\eta(x)$ satisfying conditions (a) and (b).

The functions $M(x)$, $N(x)$ are supposed to be continuous on $[x_1x_2]$.

The theorem of analytic geometry is the well known

* Compare E. H. Moore, "On a form of General Analysis with applications to linear differential equations and integral equations," Atti del IV congresso internazionale dei Mathematici, vol. 2, p. 98; and "Introduction to a form of General Analysis," in The New Haven Mathematical Colloquium, Yale University Press, New Haven, 1910.

† Compare for instance Bolza, Vorlesungen über Variationsrechnung, p. 462, footnote 1, and the references given there.

THEOREM II. *"If, in a plane and in homogeneous coordinates,*

$$U_0 \equiv A_0x + B_0y + C_0z = 0 \tag{1'}$$

is the equation of a straight line passing through the point of intersection of the two non-coinciding lines*

$$U_1 \equiv A_1x + B_1y + C_1z = 0, \quad U_2 \equiv A_2x + B_2y + C_2z = 0, \tag{2'}$$

then there exist two constants λ_1, λ_2 *such that*

$$U_0 \equiv \lambda_1 U_1 + \lambda_2 U_2."$$

§ 1. *The General Theorem.*

Let p be a general parameter† ranging over a set $\mathfrak{P}$ of elements; these elements may be any mathematical entities whatever: real or complex numbers, pairs, triples, etc., of such numbers, even infinite sets of numbers; functions of one or several variables; systems of functions; points, curves, surfaces; etc., etc.

Along with the set $\mathfrak{P}$ we consider the set Ω of all possible systems $(a_1, a_2; p_1, p_2)$ of a pair of real numbers a_1, a_2 and a pair of elements p_1, p_2 of $\mathfrak{P}$, and we suppose that a correspondence has been established by which to every element of Ω corresponds a unique element of $\mathfrak{P}$ which we denote by‡

$$F(a_1, a_2; p_1, p_2).$$

We shall then say that a real single-valued function§ $\mu(p)$ defined on $\mathfrak{P}$ is "*linear as to F*," if

$$\mu[F(a_1, a_2; p_1, p_2)] = a_1\mu(p_1) + a_2\mu(p_2) \quad \text{on } \Omega, \tag{4}$$

i. e., for every combination $(a_1, a_2; p_1, p_2)$ of Ω.

Then the following theorem holds:‖

THEOREM III. *If*

$$\mu_0(p), \ \mu_1(p), \ \cdots, \ \mu_m(p)$$

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* We may omit the word "non-coinciding" if we replace "point of intersection of" by "point or points common to."

† Compare Moore, "Introduction etc.," § 1; I use throughout this section Moore's notation.

‡ In Moore's terminology F is a "function on Ω to $\mathfrak{P}$," "Introduction etc.," § 4.

§ Compare Moore, "Introduction etc.," § 5; if $\mathfrak{A}$ denotes the set of all real numbers, $\mu(p)$ is in Moore's terminology a "function on $\mathfrak{P}$ to $\mathfrak{A}$."

‖ This generalization of Theorem I has been suggested to me by a remark in § 177 of Hadamard's Leçons sur le calcul des variations, Paris, 1910.

are $m+1$ real single-valued functions of p, defined on $\mathfrak{P}$, which satisfy the following two conditions:

A) they are linear as to F,

B) the equation

$$\mu_0(p) = 0 \tag{1''}$$

holds for every element of $\mathfrak{P}$ which satisfies simultaneously the m equations

$$\mu_1(p) = 0,\ \mu_2(p) = 0,\ \cdots,\ \mu_m(p) = 0, \tag{2''}$$

then there exist m real numbers $c_1, c_2, \cdots, c_m$, independent of p, such that

$$\mu_0(p) + c_1\mu_1(p) + \cdots + c_m\mu_m(p) = 0 \quad \text{on } \mathfrak{P}, \tag{3''}$$

i. e., for every element of $\mathfrak{P}$.

Proof: We notice first that there always exist elements of $\mathfrak{P}$ which do satisfy the m equations (2''); for $F(0, 0; p_1, p_2)$ is an element of $\mathfrak{P}$ for any two elements p_1, p_2 of $\mathfrak{P}$, and on account of A)

$$\mu_i[F(0, 0; p_1, p_2)] = 0, \quad (i = 1, 2, \cdots, m).$$

Further we observe that if we define

$$F[1, a_3;\ F(a_1, a_2; p_1, p_2), p_3] = F(a_1, a_2, a_3; p_1, p_2, p_3)$$

and generally

$$F[1, a_n;\ F(a_1, a_2, \cdots, a_{n-1}; p_1, p_2, \cdots, p_{n-1}), p_n] = F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n), \tag{5}$$

then $F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n)$ is again an element of $\mathfrak{P}$, and, if (4) is satisfied, then also

$$\mu[F(a_1, a_2, \cdots, a_n; p_1, p_2, \cdots, p_n)] = a_1\mu(p_1) + a_2\mu(p_2) + \cdots + a_n\mu(p_n). \tag{6}$$

After these preliminary remarks we distinguish two cases:

Case I: The m equations (2'') are satisfied for every p of $\mathfrak{P}$. Then according to B)

$$\mu_0(p) = 0 \text{ on } \mathfrak{P}.$$

Hence we may write

$$\mu_0(p) + 0 \cdot \mu_1(p) + 0 \cdot \mu_2(p) + \cdots + 0 \cdot \mu_m(p) = 0 \quad \text{on } \mathfrak{P},$$

and the theorem is proved with the particular values $c_1 = 0$, $c_2 = 0, \cdots, c_m = 0$.

Case II: The m equations (2″) are not all satisfied for every p of $\mathfrak{P}$.

Then there exists a definite integer n $(1 \leqq n \leqq m)$ such that in the determinant

$$\Delta = |\mu_i(p_k)| \qquad (i, k = 1, 2, \cdots, m)$$

at least one minor of degree n is different from zero for some special system $p_1, p_2, \cdots, p_m$, whereas (for $n < m$) all minors of degree $n + 1$ vanish identically, that is, for every choice of the m elements $p_1, p_2, \cdots, p_m$. In order to fix the ideas we suppose that the minor

$$\Delta_0 = |\mu_g(p_h)| \neq 0 \quad (g, h = 1, 2, \cdots, n). \tag{7}$$

Let now p be any element of $\mathfrak{P}$ and $p_1, p_2, \cdots, p_n$ the n special elements for which $\Delta_0 \neq 0$; then

$$q = F(1, a_1, a_2, \cdots, a_n; p, p_1, p_2, \cdots, p_n)$$

is an element of $\mathfrak{P}$, and according to A)

$$\mu_j(q) = \mu_j(p) + a_1\mu_j(p_1) + \cdots + a_n\mu_j(p_n) \qquad (j = 0, 1, 2, \cdots, m). \tag{8}$$

On account of (7) we can so determine $a_1, a_2, \cdots, a_n$ that

$$\mu_1(q) = 0, \ \mu_2(q) = 0, \ \cdots, \ \mu_n(q) = 0. \tag{9}$$

If $n < m$, it follows from the identical vanishing of the minors of degree $n + 1$ of the determinant Δ, p taking the place of p_{n+1}, that also

$$\mu_{n+1}(q) = 0, \ \mu_{n+2}(q) = 0, \ \cdots, \ \mu_m(q) = 0. \tag{10}$$

Hence for $n < m$ as well as for $n = m$, q is an element of $\mathfrak{P}$ which satisfies the m equations (2″) and therefore it satisfies according to B) also the equation

$$\mu_0(q) = 0. \tag{11}$$

But from the $n + 1$ equations (9) and (11) it follows, if we write the $\mu_j(q)$'s in their explicit form (8), that the determinant

$$|\mu_j(p), \ \mu_j(p_1), \ \cdots, \ \mu_j(p_n)| = 0 \qquad (j = 0, 1, 2, \cdots, n). \tag{12}$$

If now we expand this determinant according to the elements of the first column, the coefficient of $\mu_0(p)$ is the determinant Δ_0 and therefore different from zero, and this determinant as well as the remaining coefficients of the expansion is *independent of* p. Hence if we divide by Δ_0, we obtain equation (3″) with $c_{n+1}=0, c_{n+2}=0, \cdots, c_m=0$, and this equation holds on $\mathfrak{P}$, since p was *any* element of $\mathfrak{P}$. Thus our theorem is proved.*

§ 2. *Theorems I and II as Special Cases of Theorem III.*

In order to obtain Theorem I as a special case of Theorem III, we identify the set $\mathfrak{P}$ with the totality of all functions $\eta(x)$ of class C' on $[x_1x_2]$ which vanish at x_1 and x_2, and define

$$F(a_1, a_2; \eta_1, \eta_2) = a_1\eta_1 + a_2\eta_2. \tag{13}$$

If a_1, a_2 are two constants and $\eta_1(x)$, $\eta_2(x)$ two functions of $\mathfrak{P}$, $a_1\eta_1(x) + a_2\eta_2(x)$ again belongs to $\mathfrak{P}$ and the "functions"

$$\mu_j(\eta) = \int_{x_1}^{x_2} [M_j(x)\eta(x) + N_j(x)\eta'(x)]\,dx \qquad (j=0, 1, \cdots, m)$$

are "linear as to F," since

$$\mu_j(a_1\eta_1 + a_2\eta_2) = a_1\mu_j(\eta_1) + a_2\mu_j(\eta_2). \tag{14}$$

For this special choice of the set $\mathfrak{P}$, the operator F, and the functions μ_j, Theorem III becomes identical with Theorem I.

More generally we may take for $\mathfrak{P}$ the totality of all functions $\eta(x)$ of class C' on $[x_1x_2]$ which satisfy any given system of conditions provided only that these conditions are *linear*, *i. e.*, such that they are satisfied by $a_1\eta_1 + a_2\eta_2$ whenever they are satisfied by η_1 and η_2, two functions of class C' on $[x_1x_2]$. We thus obtain a generalization of Theorem I indicated by Hadamard.†

On the other hand, to obtain Theorem II as a special case of Theorem III, we identify the set $\mathfrak{P}$ with the totality of all triples $p=(x, y, z)$ formed with three independent variables x, y, z,

* I had originally thought it necessary to add to the assumptions *A*) and *B*) of the theorem the further assumption that $\Delta \neq 0$ for some system $p_1, p_2, \cdots, p_m$; I am indebted to Professor Moore for calling my attention to the fact that this assumption may be omitted, as well as for other valuable suggestions.

† loc. cit., § 176.

each ranging over all real values, and define, in Cayley's set notation,

$$F(a_1, a_2;\ p_1, p_2) = a_1(x_1, y_1, z_1) + a_2(x_2, y_2, z_2),\ i.\ e.,$$
$$\equiv (a_1x_1 + a_2x_2,\ a_1y_1 + a_2y_2,\ a_1z_1 + a_2z_2). \tag{15}$$

$F(a_1, a_2;\ p_1, p_2)$ belongs again to $\mathfrak{P}$, however the numbers a_1, a_2 and the triples $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ may be chosen.

With this definition of F, the functions

$$\mu_j(p) = A_jx + B_jy + C_jz, \quad (j = 0, 1, 2) \tag{19}$$

are "linear as to F."

If $n = 2$, there exists at least one pair of triples (x_1, y_1, z_1), (x_2, y_2, z_2) for which the determinant

$$\begin{vmatrix} A_1x_1 + B_1y_1 + C_1z_1, & A_2x_1 + B_2y_1 + C_2z_1 \\ A_1x_2 + B_1y_2 + C_1z_2, & A_2x_2 + B_2y_2 + C_2z_2 \end{vmatrix} \neq 0.$$

This means geometrically, if we interpret x, y, z as homogeneous coordinates of a point in a plane, that the two lines

$$A_1x + B_1y + C_1z = 0,\ A_2x + B_2y + C_2z = 0 \tag{20}$$

do not coincide.

Theorem III then specializes into Theorem II.

The assumption $n = 1$ leads to the trivial case alluded to on page 403, footnote *.

In like manner the corresponding theorems on pencils and bundles of planes and their generalizations to spaces of higher dimensions follow immediately as special cases from Theorem III.

THE UNIVERSITY OF CHICAGO,
February, 1910.

THE INFINITESIMAL CONTACT TRANSFORMATIONS OF MECHANICS.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, February 29, 1908.)

1. THE significance of contact transformations in the development of general dynamics and optics, appreciated to some extent by Hamilton, was first brought out explicitly by Lie.* A very thorough and elegant discussion of the whole subject, including a number of new results, has recently been given by Vessiot.† With a conservative dynamical system, defined by its potential energy U (a function of n generalized coordinates) and its kinetic energy T (a quadratic form in the n generalized velocity components), there is associated an infinitesimal contact transformation whose characteristic function W is of special type.‡ The main result of the present note may be stated as follows:

The alternant (Klammerausdruck) of the contact transformations associated with two dynamical systems, of the same number of degrees of freedom, will be a point transformation when, and only when, the expressions for the kinetic energies are either the same or differ merely by a factor.

2. For simplicity and clearness we shall confine ourselves to two degrees of freedom. The infinitesimal contact transformations are then defined by a characteristic function $W(x, y, p)$, each lineal element (x, y, p) being converted into a neighboring element $(x + \delta x, y + \delta y, p + \delta p)$ according to the standard formulas

$$(1) \quad \delta x = W_p \delta t, \quad \delta y = (p W_p - W)\delta t, \quad \delta p = -(W_x + p W_y)\delta t.$$

If the transformation is applied repeatedly to any given element, a series of ∞^1 elements is obtained, the locus of whose points is termed a path curve or trajectory. The direction of the path generated by any element is defined by the formula

* "Die infinitesimalen Berührungstransformationen der Mechanik," *Leipziger Berichte* (1889), pp. 145–153. Lie-Scheffers, Berührungstransformationen, p. 102.

† *Bulletin de la Société Mathématique de France*, vol. 34 (1906), pp. 230–269.

‡ The constant of total energy h is assumed to have a given value, so the discussion is connected with the theory of *natural families*.

$$m = \frac{\delta y}{\delta x} = p - \frac{W}{W_p}. \tag{2}$$

We shall speak of the direction m as being *transversal* * to the direction p, and shall refer to (2) as the law of transversals connected with the given transformation.

3. In the simplest case of a particle moving in a plane the kinetic energy is of the form $2T = \dot{x}^2 + \dot{y}^2$, and the associated contact transformations are of the type

$$W = \Omega(x, y)\sqrt{1 + p^2}. \tag{3}$$

Lie showed that this type is characterized geometrically by the fact that transversality reduces to orthogonality; or, what is equivalent, each point is converted into a circle of infinitesimal radius with the given point as center. We now prove that *The alternant of any two transformations of type* (3) *is a point transformation.*

For this purpose we make use of the general formula

$$W_2 = \begin{vmatrix} W_p & W_{1p} \\ W_x + pW_y & W_{1x} + pW_{1y} \end{vmatrix} - \begin{vmatrix} W & W_1 \\ W_y & W_{1y} \end{vmatrix}, \tag{4}$$

where W and W_1 are the characteristic functions of any two given transformations and W_2 is the characteristic function of their alternant (commutator, Klammerausdruck). Substituting

$$W = \Omega\sqrt{1 + p^2}, \quad W_1 = \Omega_1\sqrt{1 + p^2}, \tag{5}$$

we find

$$W_2 = p(\Omega\Omega_{1x} - \Omega_1\Omega_y) - (\Omega\Omega_{1y} - \Omega_1\Omega_y). \tag{6}$$

The linearity of this expression in p proves that it defines a point transformation; its symbol in the usual Lie notation is

$$(\Omega\Omega_{1x} - \Omega_1\Omega_x)\frac{\partial}{\partial x} + (\Omega\Omega_{1y} - \Omega_1\Omega_y)\frac{\partial}{\partial y}. \tag{6'}$$

4. The general case of two degrees of freedom is equivalent to the motion of a particle constrained to remain on an arbitrary surface, whose first quadratic form we write

* We have borrowed this term from the calculus of variations, although the idea seems quite different. The formula (2) however is precisely the transversality formula connected with the problem $\int W dx = \text{minimum}$. There are several other important analogies between the two theories.

$$ds^2 = Edx^2 + 2Fdxdy + Gdy^2. \tag{7}$$

The corresponding contact transformations are

$$W = \Omega\sqrt{E + 2Fp + Gp^2}. \tag{8}$$

The law of transversals becomes

$$E + F(m + p) + Gmp = 0, \tag{9}$$

and expresses orthogonality of directions on the surface (7). Type (8) is also characterized by the fact that each point is converted into a geodesic circle about that point as center. An important special case is dilatation, where the circles are all of equal radii; the characteristic function is then found to be

$$W = \sqrt{\frac{E + 2Fp + Gp^2}{EG - F^2}}. \tag{10}$$

Consider now two transformations

$$W=\Omega\sqrt{E+2Fp+Gp^2}, \quad W_1=\Omega_1\sqrt{E+2Fp+Gp^2}, \tag{11}$$

associated with different potentials on the same surface, and apply formula (4). It is easily verified that W_2 is linear with respect to p. The resulting point transformation is

$$\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y},$$

$$\begin{aligned} \xi &= G(\Omega\Omega_{1x} - \Omega_1\Omega_x) - F(\Omega\Omega_{1y} - \Omega_1\Omega_y), \\ \eta &= E(\Omega\Omega_{1y} - \Omega_1\Omega_y) - F(\Omega\Omega_{1x} - \Omega_1\Omega_x). \end{aligned} \tag{12}$$

Hence if two dynamical systems lead to the same expression for kinetic energy, the alternant of the associated contact transformations is a point transformation.

5. We now inquire whether this can happen when the kinetic energies differ. We may write any two of our transformations in the form

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \sqrt{\alpha_1 + 2\beta_1 p + \gamma_1 p^2}, \tag{13}$$

where the coefficients are arbitrary functions of x and y. The value of the alternant W_2 is then a complicated fractional expression. It suffices to observe that the numerator is a polynomial of the third degree in p, and the denominator is WW_1.

For a point transformation it is *necessary* that the expression shall be rational in p. Omitting the trivial case where both W and W_1 are point transformations (which has no dynamical interest), we see that the quadratics under the radicals in (13) can differ only by a factor. The work of § 4 shows that W_2 is then actually a point transformation.

Hence the alternant of the contact transformations (13) *is a point transformation when and only when* $\alpha_1 : \beta_1 : \gamma_1 = \alpha : \beta : \gamma$.

We have now completed the proof of the theorem stated in § 1.

6. The law of transversals for a transformation of the type

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2} \tag{14}$$

is of the form

$$\alpha + \beta(m + p) + \gamma m p = 0. \tag{15}$$

If this is interpreted on a proper auxiliary surface, namely, one whose length element is proportional to $\sqrt{\alpha dx^2 + 2\beta dx dy + \gamma dy^2}$, it expresses orthogonality. In the x, y plane, however, the directions p and m are conjugate with respect to a central conic. The relation (15) is of linear involutorial character, and depends only on the ratios of α, β, γ. We may therefore state the result of § 4 in geometric terms as follows:

If two contact transformations lead to the same linear involutorial law of transversals, their alternant will be a point transformation.

7. We now show that there are no other transversality laws for which an analogous result holds. In the first place we observe from (2) that if two transformations lead to the same law of transversals, the ratio of their characteristic functions is a function of x and y alone. We therefore form the alternant of W and $\lambda(x, y)W$, finding

$$W_2 = W^2\lambda_y - WW_p(\lambda_x + p\lambda_y) = S\lambda_y - \tfrac{1}{2}S_p(\lambda_x + p\lambda_y), \tag{16}$$

where S represents the square of W. The condition that W_2 shall represent a point transformation is found by placing its second derivative with respect to p equal to zero. This gives

$$(\lambda_x + p\lambda_y)S_{ppp} = 0. \tag{17}$$

The first factor vanishes only when λ is constant, a trivial case, since then the two given transformations coincide. The vanishing of the other factor shows that S must be quadratic in p, that is, W must be of the form (14).

Hence two contact transformations with the same transversality law will have a point transformation for alternant only when they are of the type

$$W = \sqrt{\alpha + 2\beta p + \gamma p^2}, \quad W_1 = \lambda \sqrt{\alpha + 2\beta p + \gamma p^2}.$$

Transversality is then expressed by a linear involutorial relation (15), *so that for each point the transversal of a given direction is the conjugate direction with respect to a conic with that point as center.*

8. A less important converse result, relating to the type considered in § 3, we state without proof. The only contact transformations which in combination with *every* transformation of type $W = \Omega \sqrt{1 + p^2}$ give a point transformation for alternant are those of the same type. The same is true even if Ω is restricted to the form $a(x^2 + y^2) + bx + cy + d$, a case of interest since then W converts circles into circles. When a vanishes the transformation belongs to the equilong class of Scheffers.

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ON AN INTEGRAL EQUATION WITH AN ADJOINED CONDITION.

BY ANNA J. PELL.

(Read before the Chicago Section of the American Mathematical Society, December 31, 1909.)

IN his doctor dissertation * Professor Cairns develops for infinitely many variables the theory of a quadratic form with an associated linear form, in order to prove the existence of solutions of the following integral equation:

$$\phi(s) = \lambda \int_a^b K(s, t)\phi(t)dt + \mu p(s), \tag{1}$$

with the adjoined condition

$$\int_a^b \phi(s)p(s)ds = 0, \tag{2}$$

where $K(s, t)$ is a given continuous symmetric function of s and t, $p(s)$ a given continuous function of s, λ and μ are parameters, and $\phi(s)$ is the function to be determined.

* "Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen," Göttingen, 1907.

A geometrical consideration suggests the possibility of transforming the equations (1) and (2) into an equivalent homogeneous integral equation with a symmetric kernel, the existence of whose solutions has already been shown by Hilbert and others. In this paper such a transformation will be carried out.

If there is a solution $\phi(s)$, the corresponding μ may be expressed

$$\mu = \frac{-\lambda \int_a^b \int_a^b K(s,t)p(s)\phi(t)ds dt}{\int_a^b [p(s)]^2 ds},$$

and the solution $\phi(s)$ satisfies the integral equation

$$\phi(s) = \lambda \int_a^b L(s,t)\phi(t)dt, \tag{3}$$

where

$$L(s,t) = K(s,t) - \frac{p(s)\int_a^b K(t,t_1)p(t_1)dt_1}{\int_a^b [p(s)]^2 ds}.$$

Conversely any solution of (3) satisfies (1) and (2). The kernel $L(s,t)$, however, is not symmetric and we cannot make any definite conclusions about the existence and character of the characteristic number λ.

Consider now the integral equation

$$\phi(s) = \lambda \int_a^b M(s,t)\phi(t)dt, \tag{4}$$

where

$$M(s,t) = L(s,t) - p(t)\frac{\int_a^b K(s,t_1)p(t_1)dt_1}{\int_a^b [p(s)]^2 ds}.$$

The kernel $M(s,t)$ is symmetric in s and t, and therefore, unless $M(s,t)$ is identically equal to zero, there exists for at least one real value of λ a solution $\phi(s)$, not identically equal to zero, of the integral equation (4).

Any solution of the equations (1) and (2) satisfies the equa-

tion (4); we must investigate under what conditions a solution of (4) is also a solution of (1) and (2).

Case I. $M(s, t) \not\equiv 0$. There exists a solution $\phi(s)$ of the equation (4); multiply this equation by $p(s)$ and integrate from a to b and we see that $\phi(s)$ satisfies the condition (2), unless the corresponding characteristic number is given by

$$\lambda_0 = \frac{-\int_a^b [p(s)]^2 ds}{\int_a^b \int_a^b K(s, t) p(s) p(t) ds dt}.$$

Suppose first that $\lambda \neq \lambda_0$; then it can easily be verified that $\phi(s)$ satisfies both (1) and (2).

If λ_0 is a characteristic number of the kernel $M(s, t)$, the function $p(s)$ is always a corresponding solution, and $p(s)$ is clearly not a solution of (1) and (2). Let $\psi(s)$ be any other solution of (4) corresponding to λ_0; then the function

$$\psi(s) - \frac{p(s) \int_a^b p(s) \psi(s) ds}{\int_a^b [p(s)]^2 ds}$$

is a solution of (4) and also of (1) and (2). Hence there exists a solution of (1) and (2) unless λ_0 is the only characteristic number of $M(s, t)$ and $p(s)$ the only corresponding solution; in this case $M(s, t)$ has the form

$$M(s, t) = k p(s) p(t), \tag{5}$$

where k is some constant not equal to zero.

Case II. $M(s, t) \equiv 0$. The equations (1) and (2) have no solution not identically equal to zero.

The final result is that the given integral equation always has a solution unless $K(s, t)$ and $p(s)$ satisfy the relation (5) * (where k may take the value zero). Further, the solutions of (1) and (2) form an orthogonal system of functions.

From the expansion theorem for the symmetric kernel $M(s, t)$ we obtain the following: any function $f(s)$ expressible in the form

* The exceptional cases are not indicated in the dissertation referred to.

$$f(s) = \int_a^b K(s, t)g(t)dt,$$

where $g(s)$ is any continuous function satisfying

$$\int_a^b p(s)g(s)ds = 0,$$

can be developed into the uniformly convergent series

$$f(s) = \frac{p(s)\int_a^b p(s)f(s)ds}{\int_a^b [p(s)]^2 ds} + \Sigma_i \phi_i(s) \int_a^b \phi_i(s)f(s)ds,$$

where $\phi_i(s)$ are the normalized solutions of (1) and (2).

That this expansion may not hold in case $g(s)$ is any continuous function (as Mr. Cairns states the theorem) is shown by the special example

$$K(s, t) = A(s)p(t) + A(t)p(s) + B(s)B(t),$$

$$A(s) \not\equiv cB(s), \quad \int_a^b A(s)p(s)ds = 0, \quad \int_a^b B(s)p(s)ds = 0,$$

$$g(s) = p(s).$$

THE UNIFICATION OF VECTORIAL NOTATIONS.

Elementi di Calcolo vettoriale con numerose Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Bologna, Nicola Zanichelli, 1909. v + 174 pp.

Omografie vettoriali con Applicazioni. By C. BURALI-FORTI and R. MARCOLONGO. Torino, G. B. Petrini, 1909. xi + 115 pp.

1. IN view of the plan that the fourth international congress of mathematicians held at Rome in 1908 should discuss the notations of vector analysis and perhaps lend the weight of its recommendation to some particular system, Burali-Forti and Marcolongo awhile ago set themselves the laudable but somewhat thankless task of collecting and editing all the historical, critical, and scientific material which might be indispensable to a proper settlement of the question by the congress, and this material they published in a series of five notes beginning in

the twenty-third volume (1907) of the *Rendiconti* of Palermo and running through several succeeding numbers and volumes. It is needless to observe that the work was accomplished with the expected accuracy. It was, however, not done with all the completeness desirable. The attention of the authors was turned almost exclusively to the minimum system most useful in mathematical physics, that is, to the questions of addition of vectors, of scalar and vector products, of differentiation with respect to a scalar, and of differentiation with respect to space (gradient of a scalar function and divergence and curl of a vector function of position). Considerable discussion was given to quaternions but the fact was fully recognized by the authors that many subjects which might rightfully have been treated were omitted — among which the linear vector function (Hamilton) or quotients and Lückenausdrücke (Grassmann) or dyadics (Gibbs) were perhaps the most noteworthy.

The authors not only followed their initial program ; they went further and themselves recommended a particular minimum system of notations essentially like any and all of those now employed but differing in the symbols selected. Thus did it appear that these strivers after unification were prone to follow the path of all unifiers and introduce still greater diversity. Seemingly the universal language of vectors, like the universal commercial language, is destined to suffer constantly new amendments at the hands of its zealots. The conception of unification as conceived in the mind of each enthusiastic unifier appears to be that he shall disagree with everybody and that everybody shall then agree with him. For the scalar product of $\mathbf{a}$ and $\mathbf{b}$ the recommendation is $\mathbf{a} \times \mathbf{b}$, for the vector product it is $\mathbf{a} \wedge \mathbf{b}$, for the gradient we have grad, and div and rot for the divergence and curl. The symbol ∇ is abolished. The suggestion $\mathbf{a} \times \mathbf{b}$ for the scalar product seems particularly infelicitous in view of the fact that this notation is in actual use for the vector product. So far as we are aware, this is the first suggestion which violently and confusingly differs from a notation which has become fairly widely established. It is true that Grassmann in some of his papers used the cross as a symbol of scalar multiplication ; but the rare and unimportant historical use of a symbol seems hardly a sufficient reason for the present adoption of the symbol in the face of actual modern usage which is tolerably popular. No such objection can be urged against the use of $\wedge$ for the vector product — that notation is altogether new.

With regard to the necessity for new systems of notations we would point out some facts. First there are already available three elaborately and consistently developed systems which represent a considerable portion of the life-work and life-thought of three great minds, namely, the systems of Hamilton, Grassmann, and Gibbs. Any of these is entirely adequate for usage in physics and in addition each represents a distinct attitude toward the science of multiple algebra. Hamilton's theory was developed from the double point of view of the theory of sets and of the exigencies of an analysis for three dimensional space. It represents algebraically the whole domain of linear associative algebra. Grassmann's work gives a general science of extensive magnitudes in any number of dimensions and contributes to algebra the important idea that a product need not be a quantity of the type of either factor. It is the prime geometric algebra. Gibbs's view seems to have been to fuse all these elements and the theory of matrices into a general science of multiple algebra (as distinguished from many individual sciences of different multiple algebras) and to construct from his general point of view a particular system especially fitted for his fellow physicists. In addition to these great scientific analyses of space we have the analysis adopted by the Encyclopedia and many present writers, especially in Germany. This system may apparently be characterized as opportunist in that it seems to have been selected largely in haphazard fashion as a sort of convenient abridged notation. From the sentimental view of scientific fitness and justice it would appear desirable to adopt, if any definitive adoption must be made, one of the systems connected with the name of a great scientist and constructed on scientific principles. From the practical point of view it might be convenient to adopt the opportunist system already so widely used. The hue and cry about the confusion due to the great diversity of notation is largely hysterical. An examination of current literature will show that there is very little diversity and that of the works currently written in vectorial notation not only a plurality but an actual and considerable majority are written in the opportunist system. The chief reason vectors are not more used is not this alleged confusion and diversity so much as it is inertia. The "news items" and "information" and "reasons" which are spread broadcast concerning vectors are about as true, about as fundamental, and about as much founded in fact as those scattered into the air about the markets

for stocks and commodities. With regard to new systems we may mention that Gibbs, with a modesty not universally in evidence, for many years refrained from publishing his system and from allowing it to be published (although he privately printed it for his own convenience and for that of his students) because he felt it did not present sufficiently original differences from existing methods. His attitude was perhaps unfortunate. The world may have been in need of the new systems; it may still be in need of new systems; this, at any rate, is the opinion of the authors whose works are under review.

Further to spread the new system and to offer to Italian scientists and students a treatise on vector analysis in their own language, the authors have written two short volumes of which the titles appear at the head of this review. It is surely a good thing that there is now at least one systematic treatment of vector analysis in Italian. The first volume, Calcolo vettoriale, takes up the matters mentioned as belonging to the minimum system; the second, Omografie vettoriali, discusses what is equivalent to the linear vector function. In their prefaces the authors state that their books differ profoundly both in method and in notation from all previous texts of recent years — differ in method because they intend to operate in an absolute manner on geometric entities, whereas ordinarily vectors and their operations are tachygraphic for coordinates; and differ in notations by the adoption of their own recommendations. It will be impossible for us to discuss these books apart from the contributions of the authors to the *Rendiconti* and to *L'Enseignement Mathématique*, which has so considerately opened its columns to an interchange of views by all interested in vectorial notations. Quotations will be made indifferently from all these sources according as they may be needed.

2. In commencing the detailed review of the two books, mention should be made of the fact that they contain not so much a pure vector analysis as a point and vector analysis with the emphasis on vectors. Thus the vector is introduced as the difference $B - A$ of two points and such equations as

$$P = A + x(B - A), \quad P = O + x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

for a line and for the position of a point are found. There is much to be said for this procedure, which introduces the origin explicitly into the analysis in addition to the system $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Its advantages and disadvantages reative to the usual pure

vector analysis will be clear to all readers. On the mathematical and logical side this usage of the authors is highly commendable; it opens the way for the consideration of Grassmann's geometrical algebra and eliminates the origin wherever unessential. Whether physicists, who have a tendency to limit their analysis to an irreducible minimum, will welcome the addition of a little point analysis is doubtful. In other respects the first three chapters give a treatment of addition and multiplication of vectors in much the usual way and with much the usual applications.

With Chapter IV there is an interesting departure from the ordinary texts. The operator i, as a quadrantal versor in a plane, is introduced by the equation * $i\mathbf{x} = \mathbf{u} \times \mathbf{x}$, where $\mathbf{x}$ is normal to $\mathbf{u}$. That this operator i, as regards its repetition, obeys the law $i^2 = -1$ of $i = \sqrt{-1}$ is pointed out. The equations

$$(i\mathbf{a}) \cdot (i\mathbf{b}) = \mathbf{a} \cdot \mathbf{b}, \quad \mathbf{a} \times (i\mathbf{b}) = \mathbf{b} \times (i\mathbf{a}),$$

however, show the reader at once that here the i is not an ordinary scalar subject to $i^2 = -1$. For if this were the case, $i\mathbf{a} \cdot i\mathbf{b}$ would be $-\mathbf{a} \cdot \mathbf{b}$, and like changes of sign would occur in other formulas. This fact taken with the fact that the operator i really depends on the vector $\mathbf{u}$ which enters into its definition seems to militate greatly against the usefulness of this chapter and the advisability of its general acceptance. It appears unfortunate to use a symbol so familiar as i in a sense which precludes its fuller treatment according to the same formal laws as it ordinarily obeys. This impression is but strengthened by the sight of the equations

$$e^{i\phi}\mathbf{a} \cdot e^{i\phi}\mathbf{b} = \mathbf{a} \cdot \mathbf{b}, \quad e^{i\phi}\mathbf{a} \cdot e^{i\psi}\mathbf{a} = \mathbf{a}^2 \cos(\psi - \phi),$$

and others of that ilk for rotations through various angles. It may well be, however, that the unfavorable impression is due merely to the unfamiliarity of the symbols, or rather, to a familiarity with them under the form of scalars subject to the laws of (complex) scalars.

After a brief mention, in Chapter V, of the differentiation of a vector with respect to a scalar the vitally important subject of differentiation with respect to space, that is, of gradient,

* Here and throughout the review the notations of the authors are translated so far as possible into the notations of Gibbs, which are probably more familiar to readers of the BULLETIN.

divergence, and curl, is treated in Chapter VI. The gradient is defined by the equation *

$$(1) \qquad dV = (\text{grad } V) \cdot dP.$$

Two remarks may be made. In the first place this definition is intrinsic or absolute, that is, devoid of reference to coordinate axes, and therein conforms to the usage we have always indicated as the best for defining ∇V. The second observation is that the authors here and elsewhere use dP, the differential of a point, instead of $d\mathbf{r}$, the differential of a vector. This has the advantage that no origin is implicated in the definition. Whether this advantage is sufficient to compensate for the slight complication of the combined point and vector analysis as against the simple vector analysis is a matter of individual opinion. The authors have a violent dislike for the symbol ∇ and not only give up its use but urge vehemently that it should be universally abandoned. This matter will be discussed later in section 4.

The defining equations for curl $\mathbf{V}$ and div $\mathbf{V}$ are given in the absolute vectorial form

$$(2) \qquad d\mathbf{r} \times \delta\mathbf{r} \cdot \text{curl } \mathbf{V} = \delta\mathbf{r} \cdot d\mathbf{V} - d\mathbf{r} \cdot \delta\mathbf{V}.$$

$$(3) \qquad \text{div } \mathbf{V} = \mathbf{a} \cdot [\text{grad } (\mathbf{a} \cdot \mathbf{V}) + \text{curl } (\mathbf{a} \times \mathbf{V})],$$

where $d\mathbf{r}$, $\delta\mathbf{r}$ are two differential displacements (written dP, δP in the text) and $d\mathbf{V}$, $\delta\mathbf{V}$ are the corresponding differentials of $\mathbf{V}$; and where the equation (3) is supposed to hold for any constant vector $\mathbf{a}$; and where finally we have seen fit for ulterior purposes to rearrange the order of the vectors in the products. These are highly ingenious definitions and have much to commend them. It will be noticed that (2) is suggested at once by the expressions that occur in the proof of Stokes's theorem by the method of variations; the definition of the divergence appears more artificial and is less intimately connected with the fundamental characteristics of div $\mathbf{V}$; both, however, are well adapted to establish some of the important formulas of the differential calculus of vectors and should therefore have the serious attention of students of the presentation of vector analysis.

It should be noticed that as defined by (1) the gradient is immediately interpretable in its physical sense as that vector

* Compare, in Gibbs's notation, $d\mathbf{r} \cdot \nabla V = dV$.

which gives in direction and magnitude the most rapid rise of V. Now the basal significance of curl $\mathbf{V}$ is found in the fact expressed by Stokes's theorem that the induction of curl $\mathbf{V}$ through a surface is equal to the integral of $\mathbf{V}$ around the boundary, and the basal significance of div $\mathbf{V}$ lies in Gauss's theorem that the integral of div $\mathbf{V}$ over a volume is equal to the induction of $\mathbf{V}$ through the bounding surface. Hence for physical reasons definitions like

$$d\mathbf{S} \cdot \operatorname{curl} \mathbf{V} = \int_{\circ} d\mathbf{r} \cdot \mathbf{V}, \tag{2'}$$

$$d\tau \operatorname{div} \mathbf{V} = \int_{\circ} d\mathbf{S} \cdot \mathbf{V}, \tag{3'}$$

where $d\mathbf{S}$ is an element of surface and $d\tau$ an element of volume, are preferable to the analytic definitions of the authors, and they have the added felicity to be immediately connected with the proofs of Stokes's and Gauss's theorems. They have the disadvantages that they are not so easily available for the proof of formulas of differential calculus and that they mix differential and integral calculus. Whether on the whole the advantages are with (2′), (3′) as against (2), (3) must remain largely a matter of personal preference.

3. It is in this same chapter that the authors reveal their remarkable discovery that the laplacian operator

$$\triangle = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is essentially different according as it is applied to a scalar or to a vector function. Upon this discovery they are especially insistent on every possible occasion.* They even go so far as to introduce different symbols $\triangle$ and $\triangle'$ for the operator according as the operand is scalar or vector. Then they are able to write

$$\begin{aligned} \triangle V &= \operatorname{div} \operatorname{grad} V, \\ \triangle' \mathbf{V} &= \operatorname{grad} \operatorname{div} \mathbf{V} - \operatorname{curl} \operatorname{curl} \mathbf{V}. \end{aligned} \tag{4}$$

Whereupon they add: You see at once what an immense difference there is between the two operators $\triangle$ and $\triangle'$ for which

* Calcolo vettoriale, p. 72; Omografie vettoriali, p. 61; and *L'Enseignement Mathématique* (1909), p. 462, from which the quotation below is taken.

authors have adopted the single symbol $\triangle_2$ because these operators have the same cartesian expression. That is no reason to use a single sign to designate very different things; on the contrary it shows once more that the systematic use of coordinates may introduce pseudo-operators which have no longer a geometric and logical character.

Now it is rather curious and decidedly regrettable that two such eminent authors, who work in a country conspicuous for its researches in the logic of mathematics and of whom one is himself illustrious for such investigations, should adduce the equations (4) as a reason for the essential difference of $\triangle$ and $\triangle'$. It merely goes to show how restricted is their point of view in the mathematics of the subject and how deficient in the physical interpretation thereof. If the authors were not so eminent and if we did not have so high a regard for most of their work in mathematics and mathematical physics, we should pass over these remarks of theirs without further comment and in full confidence that no very great number of mathematicians or physicists would fall into their way of thinking. Unfortunately that course is not open to us. The higher the sources from which vicious doctrines are promulgated, the more patient and painstaking must be their refutation. All that equations (4) do is to give two different expressions which are equivalent in the two respective cases to the laplacian operator. The second expression cannot be applied in the first case because the operators div and curl cannot be applied to a scalar; the first expression can be applied in the second case if one knows how to define the gradient of a vector function and the divergence of a linear vector function, but the authors do not give these definitions and there is no need to give them now.*

The laplacian operator is probably best known to mathematicians in connection with Laplace's equation and harmonic functions. It is a fundamental theorem on harmonic functions that the average value of the function upon the surface of a sphere is equal to the value at the center, and it is an immediate corollary that the average value throughout the sphere is equal to the value at the center. Now as the laplacian operator occurs so frequently in fundamental physical problems, it is a reasonable assumption that the operator represents some intrinsic or absolute characteristic of a field and does not depend in any other than an accidental way upon cartesian coordinates.

*See, however, (1), (2), (3) of section 7 below.

And Maxwell pointed out this characteristic which may be called the dispersion. The laplacian operator may in fact be defined by either of the equivalent intrinsic or absolute equations

$$r^2 \triangle \phi = 6(\bar{\phi} - \phi) = 10(\bar{\bar{\phi}} - \phi), \tag{5}$$

where r is the radius of an infinitesimal sphere and $\bar{\phi}$, $\bar{\bar{\phi}}$ are respectively the average value of ϕ upon the surface and throughout the volume of the sphere. This definition is comparable to (2′), (3′) for divergence and curl in that it depends on integration. But it is incomparably simpler than either of them. For they depend upon scalar products, whereas this depends only on addition as implied in averaging.

In other words the laplacian operator may and should be defined by its intrinsic properties such as are expressed in (5), and when this definition is given it appears that the operator is equally applicable and with the same significance to any quantity ϕ which satisfies the laws of addition, that is, to the elements ϕ of any linear algebra regarded as functions of position and in particular to scalar functions, vector functions, dyadic or linear-vector-function functions, and to planar vector or bivector functions. The laplacian operator is in no wise a pseudo-operator, there is no immense nor even any slight essential difference in its application to various linear fields, and there is no more reason for representing it by different symbols than there is for representing addition by different symbols. With all these facts Maxwell was familiar and so was Gibbs. The really accidental phenomenon is that the laplacian operator can be expanded in either of the forms given by (4).

4. It has been remarked that the authors do not use ∇ for grad, and it is a corollary that they should not use $\nabla\times$ and $\nabla\cdot$ for curl and divergence, which they write as rot and div. In regard to all these operators they remark that: The operators $\nabla\cdot$, $\nabla\times$ are absolute because div and rot may be defined without coordinates as in (2), (3); but ∇, on the contrary, is an essential tachygraph and the sign $\times$ or $\cdot$ which follows it is an operator which without coordinates has no meaning.* It is apparently for such reasons as these that they discard ∇ from their calculus. Now it fails to appear clear to us why grad V or ∇V is not defined by (1) quite as absolutely and just as independently of coordinates as curl $\mathbf{V}$ or $\nabla\times\mathbf{V}$ by (2) and div $\mathbf{V}$ or $\nabla\cdot\mathbf{V}$ by (3). Inasmuch as the definitions ∇V, $\nabla\cdot\mathbf{V}$,

* *L'Enseignement Mathématique* (1909), p. 466.

$\nabla \times \mathbf{V}$ are identical with those for grad V, div $\mathbf{V}$, curl $\mathbf{V}$, or may be taken so, any dependence of one set on coordinates should establish an equal dependence for the others — and we believe that in neither case is there any such dependence.

The question, however, does arise as to the availability of the detached symbols ∇, $\nabla\cdot$, $\nabla\times$, or grad, div, curl. Now in case three totally different symbols like the last set are chosen for the three types of differentiation, there can be little object in separating the operators from the operands and no analytic algorism is suggested. But if the symbols ∇, $\nabla\cdot$, $\nabla\times$ are used and if their use indicates a simple and suggestive algorism by means of which differential formulas may be remembered, then the detaching of the operators from the operand or better the introduction of the idea that the symbols ∇, $\nabla\times$, $\nabla\cdot$ should be interpreted as a combination of the operations ∇, $\times$, $\cdot$ is forcibly recommended.* For those who scorn analytic algorisms and prefer to remember a lot of distinct formulas, this argument is not impressive; but by far the larger number of persons like to have a notation which is algorismic in the sense that in itself it suggests the proper analytic transformations, and there is a very wide belief that there is much of mathematical value in a notation which has the felicity to be suggestive. The question, then, as to the preference of ∇, $\nabla\cdot$, $\nabla\times$ over grad, div, curl, is not one of dependence or independence of coordinates but one of analytic felicity.

To investigate this matter more closely, let it be granted that ∇, $\nabla\cdot$, $\nabla\times$ represent some sort of differentiation so that they may appropriately be called differentiating operators and may be expected to obey the fundamental laws of differentiation

$$(6) \qquad D(u+v) = Du + Dv, \quad D(uv) = D_u(uv) + D_v(uv),$$

where the subscripts u and v denote that u and v respectively are considered as differentiated subject to the constancy of the other. As ∇V is a vector when V is scalar, the operator ∇ may properly be called a vector operator and hence a vector differentiating operator. Next it is but natural to observe that the vector

* These remarks and those which follow, although written in the notation of Gibbs, should not be interpreted as limited to any particular system of notation; the recommendation is merely that, whatever be the notation for the scalar and vector products, the notation should be preserved and combined with ∇ or some equivalent sign of differentiation to express the divergence and curl of a vector function, while ∇ or its equivalent gives the gradient of a scalar function.

characteristic of ∇ is maintained in $\nabla \cdot \mathbf{V}$ and $\nabla \times \mathbf{V}$ provided that the $\cdot$ and $\times$ be considered in their usual general significance as operators which respectively combine two vectors into a scalar and into a vector. That there may be less chance for misconceiving our meaning, it may be stated that up to this time none of the operators is supposed to have had any cartesian interpretation and that the $\cdot$ and $\times$ cannot be considered as representing scalar and vector products of ∇ by V in the usual sense. The entire aim is to examine the formulas for differentiation with a view to determining whether or not the symbols obey the laws (6) of differentiation and the laws of scalar and vector products to an extent sufficient to warrant the statement that they are analytically suggestive.*

The formulas of differentiation as given by the authors are

$$\nabla(cV) = c\nabla V, \quad \nabla \cdot \mathbf{C} = 0, \quad \nabla \times \mathbf{C} = 0,$$
$$\nabla(U + V) = \nabla U + \nabla V, \quad \nabla \cdot (\mathbf{U} + \mathbf{V}) = \nabla \cdot \mathbf{U} + \nabla \cdot \mathbf{V},$$
$$\nabla \times (\mathbf{U} + \mathbf{V}) = \nabla \times \mathbf{U} + \nabla \times \mathbf{V}, \quad f(V)\nabla V = \nabla \int f(V) dV,$$
$$\nabla \times (V\mathbf{U}) = V\nabla \times \mathbf{U} + (\nabla V) \times \mathbf{U},$$
$$\nabla \cdot (V\mathbf{U}) = V\nabla \cdot \mathbf{U} + (\nabla V) \cdot \mathbf{U},$$
$$\nabla \cdot (\mathbf{V} \times \mathbf{U}) = \mathbf{U} \cdot \nabla \times \mathbf{V} - \mathbf{V} \cdot \nabla \times \mathbf{U}, \quad \nabla \times (\mathbf{V} \times \mathbf{U}) = ??$$

The results of the first three lines are certainly suggested by the fact that ∇ is a differentiating operator and subject to the first of equations (6). The results of the next line are similarly suggested by the second of equations (6). To show the procedure in greater detail:

$$\nabla \times (V\mathbf{U}) = \nabla_V \times (V\mathbf{U}) + \nabla_{\mathbf{U}} \times (V\mathbf{U})$$
$$= \nabla_V V \times \mathbf{U} + V\nabla_{\mathbf{U}} \times \mathbf{U}.$$

The scalar V when variable is passed out next to ∇ for differentiation, leaving the $\times$ between two vectors $\nabla V, \mathbf{U}$; and the scalar V when constant is passed out past the sign of differentiation. The formula of the last line is treated in a similar way, but the additional laws for the interchange of dot and cross in

* The fact that ∇ may be expanded as $\mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$ and that then curl $\mathbf{V}$ and div $\mathbf{V}$ may be regarded as the formal products $\nabla \times \mathbf{V}$ and $\nabla \cdot \mathbf{V}$ obtained according to the laws of multiplication would, from the present point of view, be the last instead of the first argument in justification of the notations.

a triple product or for the interchange of order in a vector product are used.

$$\nabla\cdot(\mathbf{V}\times\mathbf{U}) = \nabla_{\mathbf{V}}\cdot(\mathbf{V}\times\mathbf{U}) + \nabla_{\mathbf{U}}\cdot(\mathbf{V}\times\mathbf{U})$$
$$= \nabla_{\mathbf{V}}\times\mathbf{V}\cdot\mathbf{U} - \nabla_{\mathbf{U}}\cdot(\mathbf{U}\times\mathbf{V})$$
$$= (\nabla\times V)\cdot\mathbf{U} - (\nabla\times\mathbf{U})\cdot\mathbf{V} = \mathbf{U}\cdot\nabla\times\mathbf{V} - \mathbf{V}\cdot\nabla\times\mathbf{U}.$$

The fact that the notation suggests the result is clear.

It is desirable to go somewhat further. The authors remark * that $\nabla\times(\mathbf{V}\times\mathbf{U})$ cannot be expressed in terms of grad, div, and curl, and they use libellous language toward the operator $\mathbf{a}\cdot\nabla$ — we believe they imply that it is an essential tachygraph! Now we have a fondness for this operator because, if $\mathbf{a}$ is a unit vector, it gives the directional derivative of a vector function.† We propose therefore to inject this operator into their system in a manner quite in sympathy with their definitions of curl and divergence. Consider

$$(7)\quad \mathbf{a}\times(\nabla\times\mathbf{V}) = (\mathbf{a}\cdot\mathbf{V})\nabla - (\mathbf{a}\cdot\nabla)\mathbf{V} = \nabla(\mathbf{a}\cdot\mathbf{V}) - (\mathbf{a}\cdot\nabla)\mathbf{V},$$

where $\mathbf{a}$ is a constant vector. This formula has been obtained by expanding $\mathbf{a}\times(\nabla\times\mathbf{V})$ just as $\mathbf{a}\times(\mathbf{b}\times\mathbf{V})$ would be expanded and by the transference of ∇ in the first term to a position where it differentiates $\mathbf{V}$ as it should. Whether or not this procedure is regarded as accurate, the definition may be given that

$$(7')\qquad (\mathbf{a}\cdot\nabla)\mathbf{V} = \nabla(\mathbf{a}\cdot\mathbf{V}) - \mathbf{a}\times(\nabla\times\mathbf{V}),$$

and this definition of $(\mathbf{a}\cdot\nabla)\mathbf{V}$ is no more artificial than their definition (3) of div $\mathbf{V}$. Next let $d\mathbf{r}$ be any vector, consider $\mathbf{a}$ as the differential $\delta\mathbf{r}$, and apply the definition (2) of curl.

$$d\mathbf{r}\cdot\mathbf{a}\times(\nabla\times\mathbf{V}) = \mathbf{a}\cdot d\mathbf{V} - d\mathbf{r}\cdot\delta\mathbf{V}, \qquad \text{by (2)},$$
$$d\mathbf{r}\cdot\nabla(\mathbf{a}\cdot\mathbf{V}) = d(\mathbf{a}\cdot\mathbf{V}) = \mathbf{a}\cdot d\mathbf{V}, \qquad \text{by (1)}.$$
$$\therefore d\mathbf{r}\cdot[(\mathbf{a}\cdot\nabla)\mathbf{V}] = \mathbf{a}\cdot d\mathbf{V} - \mathbf{a}\cdot d\mathbf{V} - d\mathbf{r}\cdot\delta\mathbf{V} = d\mathbf{r}\cdot\delta\mathbf{V}.$$
$$(8)\qquad \therefore (\mathbf{a}\cdot\nabla)\mathbf{V} = \delta\mathbf{V} \quad \text{or} \quad (\delta\mathbf{r}\cdot\nabla)\mathbf{V} = \delta\mathbf{V}.$$

The last result is found by canceling the arbitrary vector $d\mathbf{r}$.

* Calcolo vettoriale, p. 68, Omografie vettoriali, p. 51.

† The directional derivative, like the laplacian operator, is a scalar operator and, like it, may be applied to the elements ϕ of any linear field when ϕ is regarded as a function of position. The definition as given in (7′) does not, however, apply in the general case, owing to the vector operations which it contains. In this respect (7′) corresponds to (4) and not to (5).

It is seen from (8) that $(d\mathbf{r}\cdot\nabla)\mathbf{V} = d\mathbf{V}$ and hence that if $\mathbf{a}$ is a unit vector $(\mathbf{a}\cdot\nabla)\mathbf{V}$ is the directional derivative of $\mathbf{V}$ in the direction $\mathbf{a}$. This result may be compared with (1). When V is scalar, ∇V is defined and $d\mathbf{r}\cdot\nabla V = dV$; the operation $(d\mathbf{r}\cdot\nabla)V$ is not defined in this case, but may naturally be defined as equal to $d\mathbf{r}\cdot\nabla V$. When $\mathbf{V}$ is a vector, $\nabla\mathbf{V}$ is not defined but $(d\mathbf{r}\cdot\nabla)\mathbf{V}$ is defined and the probability is suggested that at some future time it may become convenient to define $\nabla\mathbf{V}$ by the relation $d\mathbf{r}\cdot\nabla\mathbf{V} = (d\mathbf{r}\cdot\nabla)\mathbf{V} = d\mathbf{V}$. At any rate with this addition of the directional derivative to the system, expressions for $\nabla\times(\mathbf{V}\times\mathbf{U})$ and $\nabla(\mathbf{V}\cdot\mathbf{U})$* may be found and are identical with those suggested by the formal method.

A word may be added concerning derivatives of the second order. Here there are three important identities.

$$\nabla\cdot\nabla\times\mathbf{V} = 0, \quad \nabla\times\nabla V = 0,$$

$$\nabla\times(\nabla\times\mathbf{V}) = \nabla\nabla\cdot\mathbf{V} - \nabla\cdot\nabla\mathbf{V}.$$

Of these the first two are naturally suggested by the vector characteristics of ∇ and the third, which is equivalent to (4), tallies with the formal expansion analogous to that used in (7). Now it is by no means our intention to regard these formal derivations of the fundamental formulas of differentiation as proofs of those formulas, but merely as proofs of the statement that the notations involving ∇ are analytically suggestive and that for this reason these notations are far preferable to those like grad, div, curl. It may perfectly well be that they have disadvantages which should cause their abandonment; but these disadvantages should be clearly stated and the statements should be true and not mistaken. The only statement which we have seen and which we regard as true relative to the disadvantages of the notations ∇, $\nabla\cdot$, $\nabla\times$ is that they do not suggest the physical significance of the operations as well as grad, div, curl do. Whether this nominal infelicity outweighs the analytic suggestiveness must be left to individual opinion.†

* The result $\nabla(\mathbf{U}\cdot\mathbf{V}) = (\nabla\mathbf{U})\cdot\mathbf{V} + (\nabla\mathbf{V})\cdot\mathbf{U}$ which the formal method indicates is correct; but, as it is meaningless without definitions of $\nabla\mathbf{U}$ and $\nabla\mathbf{V}$, another form involving the directional derivative and the curl is usually given when treating only the minimum system.

† The reader will recall that at times there have been serious objections urged against replacing the S and V in the notations $S\mathfrak{ab}$ and $V\mathfrak{ab}$ for the scalar and vector products by other symbols, for the reason that no other notation so forcibly suggests which product is scalar and which is vector; nevertheless the S and V are not used much now.

5. The first half of the Calcolo vettoriale has now been covered. The second half, which is entitled applications, starts with a treatment of geometric applications, especially the tangent, normal, and binormal to curves. Here the use of vectors has decided advantages over the usual methods.

The second chapter on applications deals with fundamental theorems of integral calculus more than with applications. The formulas

$$\int \nabla \times \mathbf{V} d\tau = \int d\mathbf{S} \times \mathbf{V}, \quad \int \nabla V d\tau = \int V d\mathbf{S}. \quad \int \nabla \cdot \mathbf{V} d\tau = \int \mathbf{V} \cdot d\mathbf{S},$$

which are equivalent to an integration are given and proved. These formulas and many similar ones are all consequences of the operational equation (where any sign or no sign of multiplication may be inserted after ∇ and $d\mathbf{S}$)

$$\int\int\int d\tau \nabla(\ ) = \int\int d\mathbf{S}(\ )$$

of which the proof or for which a justification may easily be given. Green's theorem, Stokes's theorem, and the formula for the rate of change of the flux of a fluid through a surface are the other topics of the chapter. The applications to mechanics come next and are tolerably numerous — velocity and acceleration of a point and their resolutions along various directions, central motion, kinematics of a rigid body, motion of a rigid body in its plane, equilibrium of strings, motion of a rigid body in space. The selection of topics and the method of discussion are both admirable. It is here that the authors find considerable use for their operator $e^{i\phi}$ which establishes a rotation through the angle ϕ.

The applications of a more physical character follow. A short chapter on hydromechanics sets forth the derivation of the equations of motion, vortical motion, and velocity potential. It is interesting to remark how short, direct, and elegant is the treatment of these fundamental questions by vectorial methods. The elements of the theory of the equilibrium of an isotropic elastic body is given. The volume closes with applications to electromagnetism including retarded potentials, Maxwell's equations, the Poynting vector, integrals of the equations, and finally Lorentz's equations. It may therefore be seen that this volume affords a very good introduction to the elementary general theories of mathematical physics in addition to its presentation of the most important parts of vector analysis. The

work cannot fail to be of interest to physicists and to students of vectorial methods. In it may be found numerous suggestions that merit wide-spread adoption. The large amount of material which has been put into a small space without any apparent crowding or obscurity is especially noteworthy and has been accomplished largely by adherence to the program of using purely vectorial methods. Somewhat greater reference to coordinates might have made the work easier to read for those who previously were unacquainted with vectors, but certain compensating disadvantages would undoubtedly have arisen.

6. The second volume, Omografie vettoriali, is divided into three chapters which treat respectively homographies or linear transformations with a fixed origin, differentiation with respect to a point, and applications to mathematical physics. In addition there is an introduction which presents a few generalities on linear operators or transformations and an appendix which contains some added developments in the analytic theory of homographies and some applications to the differential geometry of surfaces. The linear transformation or homography is designated by a small Greek letter and the vector which results from the application of the homography α to a vector $\mathbf{u}$ is denoted by $\alpha\mathbf{u}$. The invariants of α are written as $I_3\alpha$, $I_2\alpha$, $I_1\alpha$ and are defined by the equations

$$I_3\alpha = \frac{\alpha\mathbf{u} \times \alpha\mathbf{v} \cdot \alpha\mathbf{w}}{\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}}, \quad I_2\alpha = \left[\frac{d}{dx} I_3(x + \alpha)\right]_{x=0}$$

with an additional equation involving the second derivative to define $I_1\alpha$. It is shown that $I_3\alpha$ is independent of the three independent vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ which enter into its definition and that consequently the other two invariants are also really invariants. The reader will observe that the authors have no hesitation about adding together a number x and a homography α. They regard a number, whenever convenient, as a linear transformation. The meaning of the expression $x + \alpha$ would probably be taken from the equation

$$(x + \alpha)\mathbf{u} = x\mathbf{u} + \alpha\mathbf{u}, \quad \mathbf{u} \text{ arbitrary.}$$

Now although this equation may be regarded as a justification of the usage of the authors, the question does arise as to whether or not their usage is really good. From the algebraic or matricular points of view, which they almost wholly ignore, the quantity α is an element of a quadrate algebra containing

nine units and having a modulus (often called the idemfactor), but the unit 1 is not one of the units as it is in the case of the ordinary complex numbers or in the case of quaternions, and the idemfactor, which in those cases is 1, is not 1 in the algebra of homographies α because 1 is not an element of the system. For reasons that perhaps are merely puristic we believe that it is better to regard 1 and the idemfactor as distinct and to refrain from identifying multiplication by a number with multiplication in the system. Statements like: If α, β are homographies and m is a real number, then

$$I_1(\alpha+\beta)=I_1\alpha+I_1\beta, \quad I_1(m\alpha)=mI_1\alpha, \quad I_1 m=3m,$$

seem particularly confused and infelicitous.

After a brief mention of singular homographies, of the fixed directions of a homography, and of the identical equation which a homography satisfies, the authors pass to the consideration of various types of homographies. According as one of the equations *

$$\mathbf{x}\cdot\alpha\mathbf{y}=\mathbf{y}\cdot\alpha\mathbf{x} \quad \text{or} \quad \mathbf{x}\cdot\alpha\mathbf{y}+\mathbf{y}\cdot\alpha\mathbf{x}=0$$

may be satisfied, the homography is called a dilatation or an axial homography. These correspond to what are often called self-conjugate and anti-self-conjugate or symmetric and skew-symmetric matrices or linear vector functions. The symbols D and V are introduced so that $D\alpha$ shall represent the dilatation of α or self-conjugate part of α and $V\alpha$ shall represent the vector of the axial remainder $\alpha - D\alpha$ of α. The result is that α may be written as

$$\alpha=D\alpha+V\alpha\times, \quad \text{and} \quad K\alpha=D\alpha-V\alpha\times$$

is thereupon taken as the definition of the conjugate of α. Both these equations must be regarded as operational equations and not as equations in multiple algebra. In this respect they are like $x+\alpha$. To obtain the transformation of plane areas regarded as vectors, the symbol R is introduced by the definition

$$R\alpha(\mathbf{x}\times\mathbf{y})=\alpha\mathbf{x}\times\alpha\mathbf{y}, \quad \text{and} \quad C\alpha=I_1\alpha-\alpha$$

is another definition which is reserved, however, for the appendix. There follow a large number of formulas connecting

* Readers familiar with current notations in linear associative algebra or with the notations of Hamilton, Gibbs, Cayley, or Clebsch-Aronhold for such equations as these will notice the lack of symmetry in the authors' use of the sign of multiplication and probably will regret that the idea of a post-operator was not introduced on a par with that of a pre-operator; this, however, would have involved the authors in serious notational difficulties.

the symbols I, D, V, K, R as applied to homographies (including real numbers). An additional symbol H is introduced so that $H(\mathbf{u}, \mathbf{v})$ may represent what in Gibbs's system is the dyad $\mathbf{vu}$. More relations between the symbols are given. Finally the chapter ends with a discussion of singular homographies and of versors and perversors.

From what has been said it will appear that the point of view of the authors is operational and not algebraic. Whether by taking this method and introducing these symbols they have materially added to the unity or unification of vectorial notations may be debated. Had they not been so insistent on the necessity of unification, we should say that these new methods and notations added an interesting and instructive diversity to the subject, and that the more points of view we had the better off we were both in a scientific and in a pedagogical way. Undoubtedly the best and most thorough way for anybody to learn a subject that is new and unfamiliar and unsatisfactory to him is to rewrite the subject according to his own desires; this replaces mere receptivity by original activity and lends a zest to the study. We should be happy to see everyone who is interested in vectors and who believes in their necessity adopt the authors' method and make the analysis suit himself. The adherents of unification would probably regard this proffered liberty as an invitation to license in any case other than their own. In fact although they are perfectly willing to use the operators i and $e^{i\phi}$ in a somewhat unusual way and to add numbers and homographies in a manner not in accord with the most careful practice, they are unwilling to let others use a single symbol for the laplacian operator or the notations ∇ and $\mathbf{a} \cdot \nabla$ which are of long standing and like the laplacian operator are almost universally believed to be essential operators independent of the axes of reference.

The question of the choice between the operational and algebraic treatment of strains deserves the most careful consideration if the proposition to abandon one and confine the attention to the other is seriously maintained. Linear operators are important. Owing to the researches of Volterra, Hadamard, Fréchet, and others, their importance is becoming rapidly extended to the domain of higher analysis. Hence it seems as though so much of the theory of linear operators as is applicable in general should be presented in the treatment of strains. The real question is whether the method should be applied to

strains in a detail which is not available in general and to the exclusion of the algebraic treatment. Without going into the matter of multiple algebra in general, mention may be made of the very important subject of matrices, which may now be studied so readily in the excellent presentation given by Bôcher in his Introduction to higher algebra. That there is a sort of isomorphism between matrices of the third order and strains can hardly be denied, and the isomorphism extends to matrices of higher order and strains in spaces of higher dimensions. Apart from the introduction of an arbitrary factor of proportionality, collineations in $n - 1$ dimensions may replace strains in n dimensions. The subject of matrices is naturally so presented that one who is familiar with it cannot be said to be unfamiliar with the theory of strains, especially as that theory is presented by Gibbs. And it seems only fair toward the student who for any reason learns the theory of strains to present that theory in a way which makes it conversely true that one who is familiar with the theory of strains cannot be said to be really unfamiliar with the theory of matrices. It is doubtful if the operational method is fair in this sense. Moreover although the algebraic method may not always appear quite so direct as the operational, it has the advantage of possessing an important algorism — the algorism of multiplication — which adds considerably to the ease of acquisition and offers some insight into the general domain of multiple algebra. In fine, the algebraic method seems to us to afford so much of the operational point of view as is useful for the general theory of linear operators and in addition to offer intimate points of contact with the theory of matrices and the theory of multiple algebra.*

7. The chapter on derivatives is shorter but none the less important. If u is any entity which is a function of position, the authors define the derivative

$$\frac{du}{dP} \quad \text{by} \quad \frac{du}{dP}dP = du, \tag{9}$$

and they point out that the derivative of a vector is a homo-

* We believe that a large number of writers fail to bring out clearly and perhaps even fail to realize the setting of the simple vector algebra in the wide domain of general multiple algebra. To this is attributable many a comment founded exclusively on a too restricted point of view. Whatever may be advisable pedagogically, the only scientific aspect of vector analysis is in its place in multiple algebra.

graphy and the derivative of a homography is a linear operator which converts a vector into a homography. If $\mathbf{u}$ is a vector, the derivative of $\mathbf{u}$ is not the linear vector function $\nabla\mathbf{u}$ of Gibbs; it is the conjugate $(\nabla\mathbf{u})_c$ of that function, because the authors apply the differential vector dP after the operator where Gibbs writes $d\mathbf{r}\cdot\nabla\mathbf{u} = d\mathbf{u}$ and applies it before. In like manner the derivative of a homography is not $\nabla\alpha$, but one of its five conjugates. It may be noted that, as the derivative $d\mathbf{u}/dP$ of the present authors and the derivative $\nabla\mathbf{u}$ of Gibbs are both linear vector functions, they differ from Hamilton's $\nabla\mathbf{u}$ which is a quaternion. Numerous formulas for differentiation are given. It should be remarked that Gibbs was familiar with the authors' notation for derivatives but abandoned it in favor of the notation ∇u, for reasons known probably to no one. Victor Fischer in his Vektordifferentiation und Vektorintegration resumes this notation and extends it by using a dot and cross in a manner suggested by $\nabla\cdot\mathbf{u}$ and $\nabla\times\mathbf{u}$.

The authors next proceed to define a vector grad α by

$$\text{(10)}\qquad \operatorname{grad}\alpha = \left(\frac{d\alpha}{dP}\mathbf{i}\right)\mathbf{i} + \left(\frac{d\alpha}{dP}\mathbf{j}\right)\mathbf{j} + \left(\frac{d\alpha}{dP}\mathbf{k}\right)\mathbf{k}.$$

The reader must not be so unwary as to be led to believe that the components of the vector grad α are the parentheses. The parentheses are linear vector functions which arise from applying the derivative of a homography to a vector, and the vector grad α is that which results from operating with these homographies upon the vectors after the parentheses. Although the definition of grad α is thus given in terms of the system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the vector is really independent of any set of axes. It is unfortunate that an absolute definition like those of (1), (2), (3) was not given. It looks as if the authors had temporarily fallen back to some extent into the fatal slough of tachygraphy against which they are so careful to warn us. They go on to remark that in case the homography α reduces to a number m, the noteworthy result

$$(\operatorname{grad} m)\cdot dP = dm, \quad \text{Cf. (1)},$$

arises; and with delightful ingenuousness they add that this relation does not appear to have a correlative for other homographies than numbers. Of course not! If the authors had seen fit to call the vector defined by (10) curl α or $\sqrt{\alpha}$ or anything else selected at random from the vast realm of mathe-

matical notations, they might have observed a similar lack of analogy. There is only one exception — if they had called their vector div $K\alpha$, they would have been surrounded on every side with the most persistent analogies.

Now after seeing the authors complain so much more bitterly than anybody else about the chaos of present vectorial notations and protest so vigorously against the use of a single symbol for the laplacian operator, it is difficult to say whether it is amusing or depressing to see these same authors introducing the old familiar symbol grad in a sense which they naively admit has small analogy with its former significance. Do they imagine that their present selection of notation will alleviate the chaotic condition? And do they find that the use of the same symbol for unrelated things is justifiable when it is not for things essentially identical?* If they had but listened to themselves half so attentively as they would have old hands at vector analysis listen to them, they would at least for safety's sake have used some other symbol than grad for (10) when they discovered that (10) had practically no connection with the well known grad. For this reason their choice of grad cannot be attributed to carelessness. It must be attributed to deliberateness. And this in face of the fact that various persons have used the term div in essentially the same sense as the authors use grad. Especial mention may be made of the extremely incisive and suggestive remarks of Prandtl† who treats the very subject of elasticity for which the authors are most in need of this new symbol. Even if an astute logician found no objection to the use of grad for div, it would seem as though a profound student of elasticity must feel intuitively that the forces in an elastic body arise from the divergence rather than from the slope of the fundamental homography connecting the normals to plane areas with the pressures upon them.‡

* It is interesting to quote from the authors, *L'Enseignement*, p. 466: "En conclusion, peut-on admettre, dans les mathématiques, un même nom, un même signe, pour indiquer deux choses différentes? Nous ne le croyons pas; par conséquent *nous n'avons pas suivi et nous ne suivrons jamais cette voie, qui conduit inévitablement à faire des confusions.*" The italics are ours.

† *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 13, pp. 436-449. This keen article should have the careful attention of all who are interested in vectors.

‡ The true inwardness of the incidental relation $dr.\nabla m = dm$ which arises when m is a numerical homography is seen by writing m in its proper form as mI where I is the idemfactor. Then $\nabla\cdot(mI) = \nabla m.\, I + m\nabla\cdot I$. But as I is constant, $\nabla\cdot I = 0$ and $\nabla\cdot(mI) = \nabla m$. Perhaps if the authors had not confused m and mI they would not have been led to mistake their relation grad $m\cdot dP = dm$ for an analogy with the gradient formerly defined.

Although this review is becoming lengthy we may perhaps be allowed the space to indicate what is probably the logical method of procedure in treating variable strains. With slight modifications we might use the authors' own excellent definitions (1), (2), (3) and write

$$d\mathbf{r}\cdot\nabla u = d\mathbf{r}\cdot\operatorname{grad} u = du, \tag{1}$$

$$d\mathbf{r}\times\delta\mathbf{r}\cdot\operatorname{curl} u = \delta\mathbf{r}\cdot du - dr\cdot\delta u, \tag{2}$$

$$\operatorname{div} u = \mathbf{a}\cdot[\operatorname{grad}(\mathbf{a}\cdot u) + \operatorname{curl}(\mathbf{a}\times u)], \tag{3}$$

where in (1) the entity u might be a scalar, vector, linear vector function, etc., and where in (2) and (3) the entity u may be a vector, linear vector function, etc., but not a scalar. With these definitions the reader may readily show the relation between div α and what the authors call grad α. If the methods of section 4 are adopted and extended, the definitions of curl and div become dependent on that of ∇ and the last two definitions may be suppressed. In fact these last two are introduced in the first instance so that the curl and divergence of a vector may be defined without introducing the gradient of a vector which would be a linear vector function. In like manner these definitions may be retained and the curl and divergence of a linear vector function may be defined without using the gradient of a linear vector function, which is an operator that converts vectors into linear vector functions. It is interesting to note that from the point of view of double multiplication (which must be considered in any general theory of Lückenausdrücke) the definition of the curl may be written

$$d\mathbf{r}\times\delta\mathbf{r}\cdot\nabla\times\Phi = (d\mathbf{r}\delta\mathbf{r} - \delta\mathbf{r}d\mathbf{r}):\nabla\Phi$$

and becomes formally identical with the equation

$$\alpha\times\beta\cdot\gamma\times\Phi = (\alpha\beta - \beta\alpha):\gamma\Phi, \quad (\alpha, \beta, \gamma \text{ vectors}),$$

which is a special case of a very general relation.

8. It will not be feasible to give any account of the last chapter of the Omografie, which contains applications to elasticity and to electrodynamics. There remains no space for such comments and there is very little to say. In bringing this review to a close it should be stated that the preponderating length of our adverse criticisms must not be interpreted as a wholesale condemnation of the two volumes. It has doubt-

less been noticed that the criticisms have been directed against those particular points at which, we feel confident, the authors have made incorrect statements or have unwisely abandoned fruitful algorisms and have thereby left the reader with a wrong or an unfortunately restricted point of view. There is no need to emphasize the excellent features of the work. A large number of these features are common to a considerable number of previous texts on vector analysis; many of them are new. The fact that there are so many points in which the volumes do not meet our approval is in itself evidence of the value of the books to all students of vectorial methods. In order to acquire a thorough appreciation of a subject it is necessary to examine various methods and points of view, and the restricted or even the wrong ones furnish an amount of instruction which is comparable with that furnished by those that are general and right. The present advanced state of the theory of functions of a real variable is due in no small measure to the inaccuracies or the narrow vision of earlier writers, and a considerable amount of present day instruction in this subject goes to showing that which is not true and contrasting it with that which is true.

For the benefit of vector analysis and cognate fields of mathematics we sincerely urge the general study of this work of Burali-Forti and Marcolongo and we especially recommend that each student follow their example and construct the system that pleases him most. That will be the best possible monument to the movement for unification. It will accomplish a real unity of knowledge and out of the resulting incidental diversity there will come a general and perchance not very slow elimination of the less fit and selection of the more fit. What the resulting residual system may be we will not venture to predict, but that there will be such a system fifty years hence we fully believe. And whatever that system may be it should and probably will conform to two requirements:

1° *Correct ideas* relative to vector fields,

2° *Analytic suggestiveness* of notation.

The first requirement may be fulfilled by proper teaching regardless of notation, whether vectorial, quaternionic, or cartesian; for the physicist the second is perhaps now best exemplified by the system of Gibbs, but the future may develop something preferable.

EDWIN BIDWELL WILSON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *January*, 1910.

SHORTER NOTICE.

Allgemeine Formen- und Invariantentheorie; Band I. Binäre Formen. By W. Fr. MEYER. Leipzig, Göschen (Sammlung Schubert, volume XXXIII), 1909. viii + 376 pp.

IN accordance with the general plan of the Schubert series, the treatise on invariants commences with concrete examples to lead up to general theorems, presupposes no previous knowledge of the subject, yet presents a systematic discussion which includes all the essentials of this important discipline.

The entire theory of the quadratic equation, as developed in elementary algebra, is reproduced in all detail, and the same ideas are applied to systems of equations. Anharmonic forms, involutions, Jacobians, are all explained and illustrated.

Now come linear substitutions, first translations, then inflations, and finally reciprocations. The general substitution is shown to be made up of these three, and those functions which are unchanged by the three elementary operations are therefore invariant under the general substitution. Conversely, the linear substitutions generate groups which may be classified as one, two, or three parameter groups, according to Lie.

Properties of the self corresponding elements of a general substitution and of the double elements of an involution are treated in a manner that makes this chapter a valuable appendix to a course in projective geometry.

The preceding elementary and very concrete discussion occupies 118 pages; it is followed by a chapter of 40 pages on bilinear and quadratic forms, with an introduction of the concept of the differential operator. Here again every transformation is built up from the elementary ones, the effect of each operation upon functions of the coefficients being minutely examined. As an appendix some twenty pages are now added on symbolic representation, no use of which is made in any part of the book except in the appendix at the end; in the latter the fundamental theorem is proved that every invariant can be expressed symbolically in terms of the elementary determinant forms. While this theorem is perhaps desirable for the sake of completeness, it is presented from such a different point of view that the discussion is out of harmony with the rest of the book. The treatment is so concise that the proof of the theorem will hardly be convincing to the reader, let alone any possibility of using the new method in his own work.

The second part is concerned with the differential equations connected with invariants of binary forms. Given a function $J(a_1, a_2, \cdots)$ which is invariant under the transformation $A_1(h) \equiv x_1 = \xi_1 + h_1\xi_2$, $x_2 = \xi_2$ by which

$$f(x) \equiv \sum_{i=0}^{n} n_i a_i x_1^{n-i} x_2^i$$

goes into

$$\phi(\xi) \equiv \sum_{i=1}^{n} n_i \alpha_i \xi^{n-i} \xi_2^i,$$

n_i being the binomial coefficient $n!/(n-i)!\, i!$; if $J(\alpha) = J(a)$ under $A_1(h)$ and $A_2(h)$, it is called an invariant of translation. Since α_i is a function of h as well as the a, we have

$$J(\alpha) = J(a) + h \left.\frac{dJ(\alpha)}{dh}\right]_{h=0} + \cdots,$$

from which $J'(\alpha)_{(a=0)} = 0$.

From this equation the first invariant operator

$$\nabla_1 = \sum_{i=1}^{n} i a_{i-1} \frac{\partial}{\partial a_i}$$

follows, and similarly the second,

$$\nabla_2 = \sum_{i=0}^{n} (n-i) a_{i+1} \frac{\partial}{\partial a_i}.$$

There follows a good discussion of the laws of combination of these operators and their application to a series of simultaneous forms.

The concepts of order and weight, commutators, and infinitesimal transformation are now taken up and everything reduced to the fundamental operators. There are numerous examples given for the student to work out; the algebraic work in each illustrates the idea nicely, but many could have their value increased by showing their geometric meaning, as was done in the preceding chapter.

The second chapter in the second part is concerned with relative invariants. The theorem that all the coefficients of a binary form lacking the second term are relative invariants is given an elegant proof. It is shown how to construct forms of

a given order and weight and that every invariant form for f_n is the source of a covariant for f_{n+r}. All these results are generalized to apply to a system of simultaneous forms.

The concept of an invariant is extended to apply to certain transcendental forms, including the logarithm and the elliptic integrals. By means of the former it is shown that every symmetric function of the roots can be rationally expressed in terms of the sum of the powers of the roots and a number of related theorems are derived (Waring's formulas). It is now easy to derive the expressions for the discriminant of an equation, the resultant of two such equations, and the expressions for the elementary relative invariants in terms of the roots. A second volume is in preparation which is to extend the preceding theory to ternary and quaternary forms.

To students of analytic geometry and of algebraic functions Professor Meyer's treatise will be of real assistance.

VIRGIL SNYDER.

NOTES.

THE April number (volume 11, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "The theorem of Thomson and Tait and natural families of trajectories," by EDWARD KASNER; "The introduction of ideal elements and a new definition of projective n-space," by F. W. OWENS; "The groups of classes of congruent quadratic integers with respect to a composite ideal modulus," by ARTHUR RANUM; "A simplified treatment of the regular singular point," by G. D. BIRKHOFF; "The strain of a gravitating, compressible elastic sphere," by L. M. HOSKINS.

AT the meeting of the London mathematical society held on March 10 the following papers were read: By W. F. SHEPPARD, "Forms of the remainder in the Euler-Maclaurin sum formula"; by J. W. NICHOLSON, "The scattering of light by a large conducting sphere"; by Miss H. P. HUDSON, "The 3–3 birational space transformation."

THE following papers have been read at recent meetings of the Edinburgh mathematical society. January 19: by R. SANGANA,

"Series for calculating Euler's constant and the constant in Sterling's theorem"; by D. Mc. SOMMERVILLE, "A problem in voting" and "Classification of geometries with projective matrix." February 11: by L. NAVANIENGAR, "The locus of points at which two sides of a given triangle subtend equal or supplementary angles"; by J. I. CRAIG, "Orthogonal trajectories in vectorial coordinates." March 11: by G. P. CARSLAW, "The Bolyai-Lobachevsky non-euclidean geometry; an elementary interpretation and some results which follow from it"; by R. J. T. BELL, "The locus of the lines that intersect three given lines"; by R. C. ARCHIBALD, "Note on Wallace's line."

THE Royal academy of sciences, letters and arts of Belgium announces the following prize problems for 1911:

"New investigations in the development of real or analytic functions in series of polynomials"; prize 800 francs.

"A summary of the various memoirs on systems of conics in space and a new contribution to the theory of these systems"; prize 600 francs.

Competing essays must be written in French or Flemish and sent to the secretary under the usual conditions before August 1, 1911.

THE publishing house of Gauthier-Villars in Paris announces the following additions to the series of monographs on the theory of functions edited by Professor E. BOREL: Principes de la théorie des fonctions entières de genre infini, by O. BLUMENTHAL; Leçons sur la théorie de la croissance, by A. DENJOY; Leçons sur les séries de polynomes à une variable complexe, by P. MONTEL; Leçons sur le prolongement analytique, by L. ZORETTI; L'inversion des intégrales définies, by V. VOLTERRA; Quelques principes fondamentaux de la théorie des fonctions de plusieurs variables complexes, by P. COUSIN; Leçons sur les correspondances entre variables réelles, by J. DRACH; Leçons sur la fonction $\zeta(s)$ de Riemann et son application à la théorie des nombres premiers, by H. v. KOCH.

THE following courses in advanced mathematics are offered by the various American universities during the academic year 1910–1911.

CORNELL UNIVERSITY. — By Professor J. MCMAHON: Theory of probabilities, two hours; Vector analysis, two hours.

— By Professor J. H. TANNER: Theory of equations, three hours. — By Professor J. I. HUTCHINSON: Theory of functions of a complex variable, three hours. — By Professor V. SNYDER: Descriptive geometry, three hours; Birational transformations, two hours, first term. By Professor F. R. SHARPE: Mechanics, two hours. — By Professor W. B. CARVER: Advanced calculus, three hours. — By Professor A. RANUM: Theory of groups, two hours. — By Dr. D. C. GILLESPIE: Differential geometry, two hours. — By Dr. C. F. CRAIG: Applications to mechanics and physics, two hours. — By Dr. F. W. OWENS: Differential equations, two hours. — By Dr. J. V. MCKELVEY: Advanced analytic geometry, three hours. — By Dr. L. L. SILVERMAN: Algebra of logic, two hours.

PRINCETON UNIVERSITY. — By Professor H. B. FINE: Theory of algebraic numbers, three hours, first term. — By Professor H. D. THOMPSON: Coordinate geometry, three hours. — By Professor L. P. EISENHART: Mechanics, three hours; Differential geometry, three hours. — By Professor O. VEBLEN: Linear groups and invariants, three hours, second term; Projective geometry, II, three hours, first term; Projective geometry, I, three hours. — By Professor G. D. BIRKHOFF: Differential equations, three hours; Differential equations of physics, three hours. — By Professor E. SWIFT: Theory of functions of a complex variable, I, three hours. — By Professor J. H. MCL. WEDDERBURN: Theory of functions of a complex variable, II, three hours, second term.

YALE UNIVERSITY. — By Professor J. PIERPONT: Abelian functions, two hours; Thermodynamics, two hours; Theory of functions of a complex variable, two hours; Modern analytic geometry, two hours. — By Professor P. F. SMITH: Geometrical analysis, one hour; Differential geometry, two hours; Elementary differential geometry, two hours. — By Professor E. W. BROWN: Elementary mechanics, two hours; Advanced mechanics, two hours; Advanced calculus, three hours. — By Professor W. R. LONGLEY: Calculus of variations, two hours; Potential theory and harmonic analysis, one hour. — By Dr. A. W. GRANVILLE: Elementary differential equations, one hour. — By Dr. G. M. CONWELL: Finite groups, two hours; Partial differential equations of physics, one hour. — By Dr. G. F. GUNDELFINGER: Advanced analytic geometry, two hours. — By Dr. D. D. LEIB: Transformations of space, two hours.

THE following courses in mathematics are announced for the summer semester, 1910.

UNIVERSITY OF STRASSBURG. — By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Algebra, two hours; Seminar, two hours. — By Professor F. SCHUR: Projective geometry, four hours; Theory of ordinary differential equations, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Forms and matrices, two hours; Vector analysis, three hours; Seminar, two hours. — By Professor L. v. MISES: Graphical statics, two hours; Aërial mechanics, two hours. — By Professor P. EPSTEIN: Introduction to higher mathematics, two hours. — By Professor M. SIMON: Algebraic analysis in connection with the methods of elementary arithmetic, two hours.

FOR the year 1908–1909 the list of doctorates with mathematics as the major subject conferred by German universities is as follows (the list for 1906–1908 appeared in the February BULLETIN, pages 268–274):

Berlin.

LICHTENSTEIN, L. "Zur Theorie der gewöhnlichen Differentialgleichungen und der partiellen Differentialgleichungen zweiter Ordnung. Die Lösungen als Funktionen der Randwerte und der Parameter."

Breslau.

FREUND, E. "Entwickelung willkürlicher Funktionen vermittelst meromorpher."

GOLDMAN, F. "Poncelet'sche Polygone bei Kreisen."

JOPKE, A. "Synthetische Untersuchungen über lineare Kegelschnittsysteme erster, zweiter, und dritter Stufe."

KLIEM, F. "Ueber Oerter von Treffgeraden entsprechender Strahlen in eindeutig und linear verwandten Strahlengebilden erster bis vierter Stufe."

Giessen.

LEPPER, H. "Ueber die invarianten Bildungen von Formen mit digredienten Schichten von Variabeln."

SCHMIDT, K. "Untersuchungen über Kurven dritter Ordnung im Anschluss an eine Grassmann'sche Erzeugungsweise."

WAGNER, R. "Ueber binäre bilineare und quaternäre quadratische Formen."

Göttingen.

BOLTZE, E. "Grenzschichten an Rotationskörpern in Flüssigkeiten mit kleiner Reibung."

HARR, A. "Zur Theorie der orthogonalen Funktionensysteme."

IHLENBURG, W. "Ueber die geometrischen Eigenschaften der Kreisbogenvierecke."

KOCH, H. "Ueber die praktische Anwendung der Runge-Kuttaschen Methode zur numerischen Integration von Differentialgleichungen."

SCHIMMACK, R. "Axiomatische Untersuchungen über die Vektoraddition."

SPEISER, A. "Die Theorie der binären quadratischen Formen mit Koeffizienten und Unbekannten in einem beliebigen Zahlkörper."

Greifswald.

FINKE, P. "Ueber Schaaren von ∞^5 Kurven im gewöhnlichen Raume."

HAUSSLEITER, H. "Zur Theorie der Pfaffschen Systeme."

LIER, O. "Ueber Flächenschaaren, die durch Berührungstransformation in Kurvenschaaren überführbar sind."

WERNER, A. "Ueber Systeme von drei Pfaffschen Gleichungen im Raume von fünf Dimensionen."

ZIEMKE, E. "Ueber partielle Differentialgleichungen erster Ordnung mit Integralvereinen, die als Punktmannigfaltigkeiten zweifach ausgedehnt sind."

Halle.

BOLDMANN, O. "Zur Theorie der übergeschlossenen Gelenkmechanismen."

JONAS, H. J. "Ueber W-Strahlensysteme, Flächendeformation und äquidistante Kurvenschaaren."

Jena.

GÜNTZEL, F. "Ueber Gruppierungen und Realitätsverhältnisse gewisser Punkte bei Raumkurven vierter Ordnung erster Spezies."

ROEGNER, M. "Die Steiner'sche Hypocykloide."

Kiel.

JANSEN, H. "Lückenlose Ausfüllung des R_n mit gitterförmig angeordneten n-dimensionalen Quadern."

NEUENDORFF, R. "Ueber Kreispunktpolarkurven."

Königsberg.

NEUMANN, A. "Ueber quadratische Verwandtschaften in Ebene und Raum, insbesondere Kreis- und Kugelverwandtschaften."

Leipzig.

FÖRSTER, R. "Beiträge zur specielleren Theorie der Riemannschen P-Funktionen 3ter Ordnung."

MEYER, C. "Zur Theorie des logarithmischen Potentials."

Munich.

BERWALD, L. "Krümmungseigenschaften der Brennflächen eines geradlinigen Strahlensystems und der in ihm enthaltenen Regelflächen."

BÖHM, F. "Parabolische Metrik im hyperbolischen Raum."

BURMESTER, H. "Untersuchung der wahren Hellegleichen auf der Kugel nach dem Lommel-Seeligerschen Satz."

DEBYE, P. "Der Lichtdruck auf Kugeln von beliebigem Material."

DEGENHART, H. "Ueber einige zu zwei ternären quadratischen Formen in Beziehung stehende Konnexe."

HOWLAND, L. A. "Anwendung binärer Invarianten zur Bestimmung der Wendetangenten einer Kurve dritter Ordnung."

NOETHER, F. "Ueber rollende Bewegung einer Kugel auf Rotationsflächen."

SCHMID, A. "Anwendung der Cauchy-Lipschitz'schen Methode auf lineare partielle Differentialgleichungen."

ZAPP, E. "Untersuchung eines speziellen Falles des Drei- und Vierkörperproblems."

Rostock.

JECKE, R. H. "Beiträge zur Geometrie der Bewegung."

LANGE, M. "Vereinfachte Formeln für die trigonometrische Durchrechnung optischer Systeme."

Strassburg.

MALESSA, G. "Fokale Eigenschaften korrelativer Grundgebilde."

Tübingen.

CASPER, M. "Ueber die Darstellbarkeit der homomorphen Formenschaaren durch Poincaré'sche Z-Reihen."

FRITZ, H. "Die Darstellung willkürlicher Funktionen in Anwendung auf die Statistik."

OEHLER, H. "Ueber die Gleichungssysteme, welche man aus einer Matrix variabler Elemente durch Nullsetzen der Determinanten gegebener Ordnung erhält.

Würzburg.

WIDDER, W. "Untersuchungen über die allgemeinste lineare Substitution mit vorgegebener *p*ter Potenz."

ZILLING, J. "Ueber die infinitesimale Deformation der Minimalflächen."

DURING the academic year 1908–1909 the following doctorates with mathematics as the major subject were conferred by the University of Paris:

DIENNES, P. "Essai sur les singularités des fonctions analytiques."

GAMBIER, B. "Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes."

VERONE, T. "Contribution à la théorie des ondes liquides."

MR. L. N. G. FILON, professor of pure mathematics in University College, London, and Mr. G. H. HARDY, fellow and mathematical lecturer in Trinity College, Cambridge, have been elected to membership in the Royal Society of London.

MR. G. I. TAYLOR, of Trinity College, Cambridge, was awarded a first Smith's prize for his essay on "Discontinuous motion in gases."

DR. E. MEISSNER, of the technical school of Zürich, has been promoted to a professorship of rational mechanics.

PROFESSOR H. HEEGAARD, of the military academy at Waedbeck, has been appointed professor of mathematics at the University of Copenhagen, as successor to Professor H. G. ZEUTHEN, who will retire at the close of the present academic year.

PROFESSOR A. S. CHESSIN has recently delivered lectures on the modern theory of the gyrostat at the U. S. Naval Academy and at Lafayette College.

AT Columbia University, Professor EDWARD KASNER has been promoted to a full professorship of mathematics. Mr. H. B. CURTIS has been appointed instructor in mathematics in Barnard College.

AT Cornell University, Dr. F. R. SHARPE, Dr. W. B. CARVER and Dr. A. RANUM have been promoted to assistant professorships of mathematics. Mr. W. A. HURWITZ and Mr. E. J. MILES have been appointed instructors in mathematics.

MR. A. S. HAWKESWORTH has been appointed professor of higher mathematics at the University of Pittsburg.

AT Haverford College, Professor A. H. WILSON, of the Alabama Polytechnic Institute, has been appointed associate professor of mathematics, as successor to Professor W. H. JACKSON, who returns to England.

NEW PUBLICATIONS.

(In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor, PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.)*

I. HIGHER MATHEMATICS.

BACHMANN (P.). Niedere Zahlentheorie. 2ter (Schluss-) Teil. Additive Zahlentheorie. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, X, 2.) Leipzig, Teubner, 1910. 8vo. 10 + 480 pp. Cloth. M. 17.00

DINGELDEY (F.). Sammlungen von Aufgaben zur Anwendung der Differential- und Integralrechnung. In 2 Teilen. 1ter Teil: Aufgaben zur Anwendung der Differentialrechnung. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, XXXII, 1.) Leipzig, Teubner, 1910. 8vo. 6 + 202 pp. Cloth. M. 6.00

LEBON (E.). Gaston Darboux. Biographie, bibliographie analytique des écrits. Paris, Gauthier-Villars, 1910. 8vo. 8 + 72 pp. Fr. 7.00

LORIA (G.). Spezielle algebraische und transzendente ebene Kurven-Theorie und Geschichte. Autorisierte, nach dem italienischen Manuskript bearbeitete deutsche Ausgabe von F. Schütte. (Teubners Samm. lung von Lehrbüchern auf dem Gebiete der mathematischen Wiasenschaften, V, 1.) In 2 Teilen. 1ter Band. 2te verbesserte und vermehrte Auflage. Leipzig, Teubner, 1910. 8vo. 18 + 488 pp. Cloth. M. 18.00

MASON (M.). See MOORE (E. H.).

MICKIEWICZ (B.). Das sogennante Fermatsche Problem. Krakau, 1909. 8vo. 28 pp. M. 2.00

MINKOWSKI (H.). Geometrie der Zahlen. Leipzig, Teubner, 1910. 2te Lieferung. 8vo. Pp. 8 + 241–256. M. 1.00

MOORE (E. H.), WILCZYNSKI (E. J.) and MASON (M.). The New Haven Mathematical Colloquium. Lectures delivered before members of the American Mathematical Society in connection with the summer meeting, held September 5th to 8th, 1906, under the auspices of Yale University. New Haven, Yale University Press, 1910. 8vo. 10 + 222 pp. Cloth. $3.00

NEUMANN (C.). Ueber das logarithmische Potential einer gewissen Ovalfläche. Leipzig, Teubner, 1909. M. 3.00

OLIVO (A.). Sulla soluzione dell'equazione cubica di Tartaglia: studio storico-critico. Milano, Frigerio, 1909. 16mo. 36 pp.

PENKMAYER (R.). Beweis des Satzes von Fermat. (Neubearbeitung.) München, Lindauer, 1910. 8vo. 9 pp. M. 0.50

PILGRIM (L.). Vereinfachte Behandlung der schiefwinkligen Koordinaten im Raum. Stuttgart, Metzler, 1909.

ROESER (E.). Die Verfolgungskurve auf der Kugel. (Diss.) Halle, 1909.

ROTHENBERG (S.). Geschichtliche Darstellung der Entwicklung der Theorie der singulären Lösungen totaler Differentialgleichungen von der ersten Ordnung mit zwei variablen Grössen. Leipzig, Teubner, 1908. M. 3.60

SCHUTTE (F.). See LORIA (G.).

STAUDE (O.). Analytische Geometrie des Punktpaares, des Kegelschnittes und der Fläche zweiter Ordnung. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften, XXX, 1.) 1ter Teilband. Leipzig, Teubner, 1910. 8vo. 10 + 548 pp. Cloth. M. 22.00

VOLKMANN (P.). Erkenntnistheoretische Grundzüge der Naturwissenschaften und ihre Beziehungen zum Geistesleben der Gegenwart. Allgemein wissenschaftliche Vorträge. 2te, vollständig umgearbeitete und erweiterte Auflage. (Wissenschaft und Hypothese, IX.) Leipzig, Teubner, 1910. M. 6.00

WILCZYNSKI (E. J.). See MOORE (E. H.).

II. ELEMENTARY MATHEMATICS.

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